

The air energy balance equation paradox

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Abstract

In building physics applications, mainly related to the building energy performance simulations, the master equation is the inner air energy balance equation. Historically the followed approach in writing down such master equation was to adopt the simplifying hypothesis of incompressible fluid for the air assumed dry. This hypothesis, among others, reduces the general integral energy balance equation to the usual formulation which is reported in textbooks and manuals and has been used in available energy simulation programs, such as Energy Plus, Esp-r, TRNSYS, etc. Later the moist air problem, the indoor air humidity control, has been faced just by adding a new integral balance equation: the water vapour mass balance equation. In this work it is pointed out how this "layered" historical approach leads to a paradox: an incompressible fluid is used to describe an ideal gas in an ideal gas mixture. Herewith the general integral energy balance equation is directly written in terms of moist air under ideal gas hypothesis and it is shown how, under acceptable specific hypothesis, it is possible to obtain a similar but conceptually different final formulation, which mainly differs in the transient storage term.

1. Introduction

In almost all textbooks and guides dealing with building physics, thermal loads calculations for HVAC design, buildings thermal performances and energy requirements, the generalised (i.e. unsteady) energy balance equation for the building zone is not commonly reported, meaning the indoor air integral energy conservation equation. For instance, guides like CIBSE Guide [CIBSE, 2006], the ASHRAE Handbook [ASHRAE, 2009], the Carrier Handbook [Carrier, 1965] do not report any energy balance equation, as they are focused on heating and cooling load calculations. Also

building physics textbooks do not give any detailed information on such basic equations, like [Hagentofl, 2001], [Hens, 2010], and just a few lines are normally present in buildings' heating and cooling textbooks without any explanation, like [Kreider, 2002], [Balaras, 1996]. Surprisingly also books devoted to energy performance modelling and simulation appear to be insufficient in analysing and describing that item, like [Underwood, 2004] and [Clarke, 2001]. Almost all those texts and the scientific literature dealing with energy transfer and performances in building (just as an example (Abuku, 2009)) use as an "energy" balance equation or "heat balance" or "sensible energy" balance equation for a building zone the following equation:

$$\begin{aligned} \rho_a c_p V_a \frac{dT_a}{dt} = & \sum_{j=1}^{N_{Surfaces}} A_j h_{cv_j} (T_{S_j} - T_a) + \\ & + \sum_{j=1}^{N_{zone}} \dot{m}_j c_p \cdot (T_{a_j} - T_a) + \dot{m}_{inf} c_p \cdot (T_{a,ext} - T_a) + \\ & + \dot{Q}_I^{(CV)} + \dot{Q}_{SYS}^{(CV)} \end{aligned} \quad (1)$$

where the first term represents the indoor air thermal energy increase or decrease due to zone surfaces-to-fluid convective heat flows, inter-zone air mixing, heat transfer due to infiltration of outside air, convective internal load and air H&C system input to the zone.

Few of them then add the "latent energy" balance equation, which is always written as humidity ratio balance equation (i.e. water vapour mass balance equation), as follows:

$$\begin{aligned} \rho_a V_a \frac{dx_a}{dt} = & \sum_{j=1}^{N_{Surfaces}} A_j h_{m_j} (x_{s_j} - x_a) + \sum_{j=1}^{N_{zone}} \dot{m}_j \cdot (x_{a_j} - x_a) + \\ & + \dot{m}_{inf} \cdot (x_{a,ext} - x_a) + \dot{M}_I^{(x)} + \dot{m}_{SYS} \cdot (x_{SYS} - x_a) \end{aligned} \quad (2)$$

where the first term represents the indoor air humidity increase or decrease due to zone

surfaces-to-fluid convective moisture flows, inter-zone air flows, humidity transfer due to infiltration of outside air, humidity internal load and air H&C system humidity net contribution to the zone.

Finally the three most known and used computer simulation tools for building energy performance assessment, Energy Plus (Energy Plus, 2012), ESP-r (Clarke, 2001; Aissan, 2005) and TRNSYS (TRNSYS, 2007), are just employing equation (1) and equation (2) for each zone air node without any reference to a deep explanation on their consistency with the simulated problem: moist air.

Analysing equations (1) and (2) it is evident that the following hypotheses are recalled:

- dry air as incompressible inviscid fluid in equation (1);
- moist air as ideal gas mixture in equation (2) as usually assumed to allow the writing of the water vapour mass balance in terms of humidity ratio and dry air mass flow rate.

Are they consistent with each other? It seems not. This contradiction may be, apparently, solved just observing that equation (1) can also be valid for an ideal gas which undergoes an isobaric process. Then what about the dry air mass conservation equation, which is embedded into equation (1) and (2) and implicitly assumes again dry air as incompressible fluid?

To solve this contradiction, the energy balance equation for a moist air volume (zone) will be directly derived in terms of moist air enthalpy and internal energy together with the mass conservation equation for both components, dry air and water vapour.

2. Basic Conservation Principles

To state clearly all necessary hypotheses to obtain a reliable energy balance equation for the zone indoor moist air, the best procedure is to start from the basic conservation principles:

- mass conservation principle;
- momentum conservation principle;
- energy conservation principle;

adding the necessary phenomenological or constitutive laws and the state equations for the considered phenomena and system.

Following the Eulerian approach (i.e. control volume approach), the integral form of the aforementioned conservation principles are given by:

$$\frac{dm^{(\alpha)}}{dt} = \sum_{j=1}^{N_{\phi}} \Phi_{cv_j}^{(m^{(\alpha)})} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(\alpha)} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(m^{(\alpha)})} \quad (3)$$

$$\frac{d\mathbf{P}}{dt} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(\mathbf{P})} + \mathbf{F}_s + \mathbf{F}_b \quad (4)$$

$$\frac{dE}{dt} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(E)} + \Phi_T + \dot{W}_s + \dot{W}_b \quad (5)$$

where equation (3) has been written for a generic phase α of a material component, and thus the source term $\dot{M}_k^{(\alpha)}$ is just representing mass exchange between different phases. Because the source terms we deal with, $M_k^{(\alpha)}$ is not a point source distributed inside the whole control volume, these source terms are included into extra advective flows $\Phi_{cv_j}^{(m^{(\alpha)})}$. This is possible because in our cases we usually have a clear separation surface between the source and the fluid; thus the mass “produced” by the sources can be simply represented through a flow that crosses the source-bounding surface (internal CV boundaries). Herewith the advective term is used instead of convective to stress the mass transfer due to bulk flow only (i.e. without mass diffusion). In the vectorial equation (4) \mathbf{P} is the momentum vector and \mathbf{F}_s and \mathbf{F}_b are respectively the superficial forces resultant vector and the body forces resultant vector, while the $\Phi_{cv_j}^{(\mathbf{P})}$ terms represent the momentum advection through the CV boundaries. In equation (5) E is the CV total energy, Φ_T is the conductive heat transfer from the bounding material surfaces to the fluid and fluid-to-fluid through the inlet/outlet mass flow surfaces, \dot{W}_s and \dot{W}_b respectively the power of surface forces and body forces, while the $\Phi_{cv_j}^{(E)}$ terms are the advective energy flows through the CV boundaries.

These integral principles can also be expressed in terms of local quantities as:

$$\frac{d}{dt} \int_{CV} \rho^{(\alpha)} dV = - \sum_{j=1}^{N_{\phi+\sigma}} \int_{S_{CV,j}} \rho^{(\alpha)} \mathbf{v} \cdot \mathbf{n} dS \quad (6)$$

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho \mathbf{v} dV = & - \sum_{j=1}^{N_{\phi+\sigma}} \int_{S_{CV,j}} (\rho \mathbf{v}) \mathbf{v} \cdot \mathbf{n} dS + \\ & + \int_{CV} \mathbf{f}_s dV + \int_{CV} \rho \mathbf{f}_b dV \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho e dV = & - \sum_{j=1}^{N_{\phi+\sigma}} \int_{S_{CV,j}} (\rho e) \mathbf{v} \cdot \mathbf{n} dS + \\ & - \sum_{i=1}^{N_s} \int_{S_{CV,j}} \boldsymbol{\phi}_T \cdot \mathbf{n} dS + \\ & + \int_{S_{CV}} \mathbf{f}_s \cdot \mathbf{v} dV + \int_{CV} \rho \mathbf{f}_b \cdot \mathbf{v} dV \end{aligned} \quad (8)$$

where the local quantities are the volumetric mass ρ , the velocity vector \mathbf{v} , the specific force vectors by unit of surface \mathbf{f}_s and by unit of mass \mathbf{f}_b , the specific total energy e , the heat flow density vector $\boldsymbol{\phi}$.

To derive the local conservation principles the structure of the specific force vectors by unit of surface has to be consider, i.e.:

$$\mathbf{f}_s = \mathbf{T} \cdot \mathbf{n} \quad (9)$$

where \mathbf{T} is the symmetric stress tensor, obtaining:

$$\frac{\partial \rho^{(\alpha)}}{\partial t} + \text{div}(\rho^{(\alpha)} \mathbf{v}) = 0 \quad (10)$$

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \text{div}(\rho \mathbf{v} \mathbf{v}) - \text{div}(\mathbf{T}) - \rho \mathbf{f}_b = 0 \quad (11)$$

$$\frac{\partial}{\partial t} (\rho e) + \text{div}(\rho e \mathbf{v}) + \text{div}(\boldsymbol{\phi}_T) - \text{div}(\mathbf{T} \cdot \mathbf{v}) - \rho \mathbf{f}_b \cdot \mathbf{v} = 0 \quad (12)$$

where $\mathbf{v} \mathbf{v}$ is the dyadic product¹ of vector \mathbf{v} with itself.

In equations (5), (8) and (12), assessing the energy conservation, the total and the specific total energy is used, defined as the sum of internal and kinetic energy:

$$E = U + E_k \quad ; \quad e = u + \frac{1}{2} v^2 \quad (13)$$

Substituting equation (13), second term, into equation (12) we obtain:

$$\begin{aligned} \frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} \right) + \text{div}(\rho u \mathbf{v}) + \text{div} \left(\rho \frac{v^2}{2} \mathbf{v} \right) + \\ + \text{div}(\boldsymbol{\phi}_T) - \text{div}(\mathbf{T} \cdot \mathbf{v}) - \rho \mathbf{f}_b \cdot \mathbf{v} = 0 \end{aligned} \quad (14)$$

which suggests the possibility to split the energy conservation principle into two balance equations, one related to the internal energy, the other to the kinetic energy.

2.1 Local Balance Equations

To achieve such a result allows us to have the dot product of the velocity vector \mathbf{v} and the local momentum conservation equation (11); this will result in the following balance equation (no longer conservation principle):

$$\frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} \right) + \text{div} \left(\rho \frac{v^2}{2} \mathbf{v} \right) - \mathbf{v} \cdot \text{div}(\mathbf{T}) - \rho \mathbf{v} \cdot \mathbf{f}_b = 0 \quad (15)$$

which is the *local Mechanic Energy Balance Equation*.

Subtracting equation (15) from equation (14) the *local Internal Energy Balance Equation* is derived:

$$\frac{\partial}{\partial t} (\rho u) + \text{div}(\rho u \mathbf{v}) + \text{div}(\boldsymbol{\phi}_T) - \text{tr}(\mathbf{T} \cdot \nabla \mathbf{v}) = 0 \quad (16)$$

where $\text{tr}()$ is the trace operator² as defined in linear algebra, and it comes out from the following differentiation rule:

$$\text{div}(\mathbf{T} \cdot \mathbf{v}) = \mathbf{v} \cdot \text{div}(\mathbf{T}) + \text{tr}(\mathbf{T} \cdot \nabla \mathbf{v}) \quad (17)$$

Using such a rule an alternative writing of equation (15) is possible as:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} \right) + \text{div} \left(\rho \frac{v^2}{2} \mathbf{v} \right) + \text{tr}(\mathbf{T} \cdot \nabla \mathbf{v}) + \\ - \text{div}(\mathbf{T} \cdot \mathbf{v}) - \rho \mathbf{v} \cdot \mathbf{f}_b = 0 \end{aligned} \quad (18)$$

which evidences, when compared with equation (16), the coupling term between mechanic energy and internal energy.

¹ the dyadic product of two vectors \mathbf{a} and \mathbf{b} is denoted by the juxtaposition $\mathbf{a} \mathbf{b}$, and its result, called dyad, is a second order tensor.

² the trace of an n-by-n square matrix A is defined to be the sum of the elements on the main diagonal, and the dot product between the two second order tensors, the stress tensor and the dyad constituted of the nabla operator and the velocity vector, is equivalent to a 3-by-3 square matrix and is a scalar quantity.

To clarify the meaning of such a coupling term it is convenient to introduce the usual splitting of the stress tensor in isotropic and anisotropic components,

$$\mathbf{T} = -P\mathbf{I} + \mathbf{D} \quad (19)$$

where P is the thermodynamic pressure and \mathbf{D} the deviatoric stress tensor.

Substituting equation (19) into equation (18) and (16) it follows:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} \right) + \operatorname{div} \left(\rho \frac{v^2}{2} \mathbf{v} \right) - P \operatorname{div}(\mathbf{v}) + \operatorname{tr}(\mathbf{D} \cdot \nabla \mathbf{v}) + \\ + \operatorname{div}(P\mathbf{v}) - \operatorname{div}(\mathbf{D} \cdot \mathbf{v}) - \rho \mathbf{v} \cdot \mathbf{f}_b = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \mathbf{v}) + \operatorname{div}(\boldsymbol{\varphi}_T) + P \operatorname{div}(\mathbf{v}) + \\ - \operatorname{tr}(\mathbf{D} \cdot \nabla \mathbf{v}) = 0 \end{aligned} \quad (21)$$

Defining

$$\phi_D \equiv \operatorname{tr}(\mathbf{D} \cdot \nabla \mathbf{v}) \geq 0 \quad (22)$$

$$\dot{w}_{R,c/e} \equiv -P \operatorname{div}(\mathbf{v}) \quad (23)$$

to be respectively the *local dissipation function* and the *local reversible compression/expansion power*, equations (20) and (21) can be more significantly rewritten as:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\rho \frac{v^2}{2} \right) + \operatorname{div} \left(\rho \frac{v^2}{2} \mathbf{v} \right) + \dot{w}_{R,c/e} + \phi_D + \\ - \dot{w}_s - \dot{w}_b = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}(\rho u \mathbf{v}) + \operatorname{div}(\boldsymbol{\varphi}_T) - \dot{w}_{R,c/e} - \phi_D = 0 \end{aligned} \quad (25)$$

that clearly shows reversible and irreversible conversion of local mechanical energy into internal energy.

The local reversible compression/expansion power, using the local mass conservation equation (10), can then be rewritten as:

$$\dot{w}_{R,c/e} \equiv \frac{P}{\rho} \left(\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right) = \frac{P}{\rho} \cdot \frac{D\rho}{Dt} \quad (26)$$

where the last term is the material derivative³ of the mass by volume.

³ Material derivative is defined as $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$

2.2 Enthalpy Local Balance Equations

It is useful at this stage to introduce a new thermodynamic quantity, the specific enthalpy:

$$h = u + P/\rho \quad (27)$$

Substituting that into equation (25) and using equation (22) and (23) leads to a new useful equation, the *local enthalpy balance equation*:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho h) + \operatorname{div}(\rho h \mathbf{v}) + \operatorname{div}(\boldsymbol{\varphi}_T) + \\ - \frac{\partial}{\partial t}(P) - \mathbf{v} \cdot \nabla P - \operatorname{tr}(\mathbf{D} \cdot \nabla \mathbf{v}) = 0 \end{aligned} \quad (28)$$

which, recalling the definition of material derivative and using the definition given through equation (22), can be rewritten as:

$$\frac{\partial}{\partial t}(\rho h) + \operatorname{div}(\rho h \mathbf{v}) + \operatorname{div}(\boldsymbol{\varphi}_T) - \frac{DP}{Dt} - \phi_D = 0 \quad (29)$$

Integral balance equations

From the local balance equations, the following integral balance equations can be finally derived:

$$\frac{dE_k}{dt} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(E_k)} - \dot{W}_{R,c/e} - \Phi_D + \dot{W}_s + \dot{W}_b \quad (30)$$

$$\frac{dU}{dt} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(U)} + \Phi_T + \dot{W}_{R,c/e} + \Phi_D \quad (31)$$

$$\frac{dH}{dt} = \sum_{j=1}^{N_{\phi+\sigma}} \Phi_{cv_j}^{(H)} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV + \Phi_D \quad (32)$$

3. Constitutive Laws

The conservation principles alone do not allow us to solve any problem, and the specific nature of the problem and of the matter constituting the system have to be introduced via constitutive laws.

3.1 Newtonian Stokes Compressible Fluid

The constitutive equation for the deviatoric stress tensor of a Newtonian compressible fluid, applying the Stokes' relation, is the following:

$$\mathbf{D} = \mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) - \frac{2}{3} \mu \operatorname{div}(\mathbf{v}) \mathbf{I} \quad (33)$$

Equation (33) shows that the deviatoric stress tensor is a function of the dynamic viscosity and of

the gradient and the divergence of the velocity vector.

3.2 Fourier Heat Flux Density

The conductive heat transfer inside a fluid can be described using the Fourier postulate:

$$\Phi_T = -\lambda \cdot \nabla T \quad (34)$$

and for an isotropic medium:

$$\Phi_T = -\lambda \nabla T \quad (35)$$

3.3 Ideal Gas Laws

The constitutive equations for an ideal gas are:

$$u = u_0 + c_v(T - T_0) \quad ; \quad c_v = \text{const} \quad (36)$$

$$h = h_0 + c_p(T - T_0) \quad ; \quad c_p = \text{const} \quad (37)$$

$$\frac{P}{\rho} = \frac{R}{M_m} T \quad (38)$$

3.4 Moist Air Constitutive Equations

Almost all psychrometric quantities are based on the Ideal Gas Laws applied to an ideal gas mixture constituted of dry air and water vapour, both considered ideal gasses (i.e. compressible fluids). There are three main relationships based on this constitutive hypotheses, the humidity by mass definition, the moist air specific internal energy and enthalpy:

$$x \equiv \frac{dm^{(wv)}}{dm^{(da)}} = \frac{dm^{(wv)}}{dV} \cdot \frac{dV}{dm^{(da)}} = \frac{\rho^{(wv)}}{\rho^{(da)}} \quad (39)$$

$$u^{(ma)} = u^{(da)} + x \cdot u^{(wv)} \quad (40)$$

$$h^{(ma)} = h^{(da)} + x \cdot h^{(wv)} \quad (41)$$

all relatives to the unit of mass of dry air.

It is very common to fix the reference point 0 for the dry air specific enthalpy and the water vapour specific enthalpy as follows:

$$h_0^{(da)} \equiv 0 \quad \text{at} \quad T = 273.15 \text{ K} \quad (42)$$

$$h_0^{(w)} \equiv h_{TP}^{(lw)} \equiv 0 \quad \text{i.e. at} \quad T = 273.16 \text{ K}$$

i.e. zero degree Celsius for the first and the water triple point for the second (0.01 °C)

The water vapour specific enthalpy is usually expressed (ASHRAE, 2009; Kreider, 2002) in terms of temperature considering the phase change process from saturated liquid to saturated vapour along the triple point isothermobaric evaporation path and then the superheated vapour performing

as an ideal gas. This approach results in the following constitutive equation for the water vapour specific enthalpy:

$$h^{(wv)}(T) = h_{TP}^{(lw)} + \Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)}(T - T_{TP}) \quad (43)$$

where $\Delta h_{l \rightarrow w}^{(wTP)}$ is the phase change enthalpy.

Combining equation (37) for dry air with equations (42) and (43) into equation (41), the constitutive law for moist air specific enthalpy is obtained as:

$$h^{(ma)}(T) = c_p^{(da)}(T - 373.15) + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)}(T - 373.16)] \quad (44)$$

which can be rewritten, with a negligible approximation, using the Celsius temperature, ϑ , as:

$$h^{(ma)}(T) = c_p^{(da)} \vartheta + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \vartheta] \quad (45)$$

Similarly, analogous expression can be derived for the specific internal energy of moist air:

$$u^{(ma)}(T) = c_v^{(da)} \vartheta + x \cdot [\Delta u_{l \rightarrow w}^{(wTP)} + c_v^{(w)} \vartheta] \quad (46)$$

4. Additional Relationships

4.1 Newton's Law of Cooling

The convective heat transfer between a solid surface and a moving fluid, conductive at the interface, can be described at the integral level as:

$$\Phi_T = \sum_{i=1}^{N_s} h_{cv,i} (T_{s_i} - T_{fr}) \quad (47)$$

5. Moist Air Balance Equations

The control volume (i.e. indoor air of the building zone) is filled with moist air; thus in the following the conservation principles in terms of integral equations and the integral balance equations will be used together with the moist air constitutive equations.

5.1 Dry Air Mass Conservation

The integral mass conservation principle for dry air is, from equation (3),:

$$\frac{dm^{(da)}}{dt} = \sum_{j=1}^{N_b} \Phi_{cv_j}^{(m^{(da)})} \quad (48)$$

or, from equation (6):

$$\frac{d}{dt} \int_{CV} \rho^{(da)} dV = - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(da)} \mathbf{v} \cdot \mathbf{n} dS \quad (49)$$

Applying the Integral Mean Value Theorem on both sides of equation (49), keeping constant the CV, we get:

$$V_{CV} \frac{d}{dt} \langle \rho^{(da)} \rangle_{CV} = \pm \sum_{j=1}^{N_\phi} A_{S_{CV,j}} \langle \rho^{(da)} \mathbf{v}_n \rangle_{S_{CV,j}} \quad (50)$$

where the terms on the right side are usually expressed in terms of mass flow rate (always positive quantities) splitting the sum in inlet and outlet flows:

$$\pm \sum_{j=1}^{N_\phi} A_{S_{CV,j}} \langle \rho^{(da)} \mathbf{v}_n \rangle_{S_{CV,j}} = \sum_{j=1}^{N_{\phi,in}} \dot{m}_{in_j}^{(da)} - \sum_{j=1}^{N_{\phi,out}} \dot{m}_{out_j}^{(da)} \quad (51)$$

Thus equation (50) can be written as:

$$V_{CV} \frac{d}{dt} \langle \rho^{(da)} \rangle_{CV} = \sum_{j=1}^{N_{\phi,in}} \dot{m}_{in_j}^{(da)} - \sum_{j=1}^{N_{\phi,out}} \dot{m}_{out_j}^{(da)} \quad (52)$$

5.2 Water Vapour Mass Balance

From equation (3), with the source term made explicit, the integral mass conservation principle applied only to the vapour phase of water is:

$$\frac{dm^{(wv)}}{dt} = \sum_{j=1}^{N_\phi} \Phi_{cv_j}^{(m^{(wv)})} + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (53)$$

or, from equation (6):

$$\frac{d}{dt} \int_{CV} \rho^{(wv)} dV = - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(wv)} \mathbf{v} \cdot \mathbf{n} dS + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (54)$$

Applying the Integral Mean Value Theorem on both sides of equation (54), keeping constant the CV, we obtain:

$$V_{CV} \frac{d}{dt} \langle \rho^{(wv)} \rangle_{CV} = \pm \sum_{j=1}^{N_\phi} A_{S_{CV,j}} \langle \rho^{(wv)} \mathbf{v}_n \rangle_{S_{CV,j}} + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (55)$$

which can be rewritten as

$$V_{CV} \frac{d}{dt} \langle \rho^{(wv)} \rangle_{CV} = \sum_{j=1}^{N_{\phi,in}} \dot{m}_{in_j}^{(wv)} - \sum_{j=1}^{N_{\phi,out}} \dot{m}_{out_j}^{(wv)} + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (56)$$

Applying the constitutive equation (39) to equation (55) it follows:

$$V_{CV} \frac{d}{dt} \langle x \cdot \rho^{(da)} \rangle_{CV} = \pm \sum_{j=1}^{N_\phi} A_{S_{CV,j}} \langle x \cdot \rho^{(da)} \mathbf{v}_n \rangle_{S_{CV,j}} + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (57)$$

which can be written as

$$V_{CV} \frac{d}{dt} \langle x \cdot \rho^{(da)} \rangle_{CV} = \sum_{j=1}^{N_{\phi,in}} \dot{m}_{in_j}^{(da)} \langle x \rangle_{S_{in_j}} - \sum_{j=1}^{N_{\phi,out}} \dot{m}_{out_j}^{(da)} \langle x \rangle_{S_{out_j}} + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} \quad (58)$$

only under the following hypothesis:

$$[\mathbf{H.0}] \langle a \cdot b \rangle = \langle a \rangle \cdot \langle b \rangle + \text{cov}(a, b) ; \text{cov}(a, b) \cong 0$$

i.e. there is no or a weak cross correlation between a and b over the average operator field (covariance ≈ 0), in this case between the dry air mass flow rate and the air humidity over the inlet/outlet cross section.

5.3 Moist Air Enthalpy Balance

The integral enthalpy balance expressed by equation (32) can be written, in terms of specific quantities and splitting the enthalpy flows due to pure advection from those related to vapour sources, as:

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho^{(da)} h^{(ma)} dV = & - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(da)} h^{(ma)} \mathbf{v} \cdot \mathbf{n} dS + \\ & + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} h_k^{(wv)} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV + \Phi_D \end{aligned} \quad (59)$$

and introducing the constitutive law for moist air specific enthalpy given by equation (45), it turns out in:

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho^{(da)} \{ c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(w_{TP})} + c_p^{(w)} \mathcal{G}] \} dV = \\ = - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(da)} \{ c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(w_{TP})} + c_p^{(w)} \mathcal{G}] \} \mathbf{v} \cdot \mathbf{n} dS + \\ + \sum_{k=1}^{N_\sigma} \dot{M}_k^{(wv)} h_k^{(wv)} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV + \Phi_D \end{aligned} \quad (60)$$

Applying the Integral Mean Value Theorem on the left side of equation (60), keeping constant the CV, we obtain:

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho^{(da)} \{c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \mathcal{G}]\} dV = \\ = V_{CV} \frac{d}{dt} \left\{ c_p^{(da)} \langle \rho^{(da)} \mathcal{G} \rangle_{CV} + \Delta h_{l \rightarrow w}^{(wTP)} \langle \rho^{(da)} x \rangle_{CV} + \right. \\ \left. + c_p^{(w)} \langle \rho^{(da)} x \cdot \mathcal{G} \rangle_{CV} \right\} \end{aligned} \quad (61)$$

and applying hypothesis [H.0] it becomes:

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho^{(da)} \{c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \mathcal{G}]\} dV = \\ \equiv V_{CV} \left[c_p^{(da)} + \langle x \rangle_{CV} c_p^{(w)} \right] \frac{d}{dt} \langle \rho^{(da)} \mathcal{G} \rangle_{CV} + \\ + V_{CV} \left[\Delta h_{l \rightarrow w}^{(wTP)} \frac{d}{dt} \langle \rho^{(da)} x \rangle_{CV} + \right. \\ \left. + c_p^{(w)} \langle \rho^{(da)} \mathcal{G} \rangle_{CV} \frac{d}{dt} \langle x \rangle_{CV} \right] \end{aligned} \quad (62)$$

where

$$c_p^{(ma)} \{ \langle x \rangle_{CV} \} \equiv [c_p^{(da)} + \langle x \rangle_{CV} c_p^{(w)}] \quad (63)$$

is the moist air constant pressure specific thermal capacity.

Applying the Integral Mean Value Theorem on the convective enthalpy flow terms of the right side of equation (60), keeping constant the CV, we obtain:

$$\begin{aligned} - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(da)} \{c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \mathcal{G}]\} \mathbf{v} \cdot \mathbf{n} dS = \\ \pm \sum_{j=1}^{N_\phi} A_{S_{CV,j}} \left\{ c_p^{(da)} \langle \rho^{(da)} v_n \mathcal{G} \rangle_{S_{CV,j}} + \right. \\ \left. + \Delta h_{l \rightarrow w}^{(wTP)} \langle \rho^{(da)} v_n x \rangle_{S_{CV,j}} + c_p^{(w)} \langle \rho^{(da)} v_n x \mathcal{G} \rangle_{S_{CV,j}} \right\} \end{aligned} \quad (64)$$

Applying hypothesis [H.0], i.e. assuming that:

$$\begin{aligned} \langle \rho^{(da)} v_n \mathcal{G} \rangle_{S_{CV,j}} &\equiv \langle \rho^{(da)} v_n \rangle_{S_{CV,j}} \langle \mathcal{G} \rangle_{S_{CV,j}} \\ \langle \rho^{(da)} v_n x \rangle_{S_{CV,j}} &\equiv \langle \rho^{(da)} v_n \rangle_{S_{CV,j}} \langle x \rangle_{S_{CV,j}} \\ \langle \rho^{(da)} v_n x \mathcal{G} \rangle_{S_{CV,j}} &\equiv \langle \rho^{(da)} v_n \rangle_{S_{CV,j}} \langle x \rangle_{S_{CV,j}} \langle \mathcal{G} \rangle_{S_{CV,j}} \end{aligned} \quad (65)$$

the enthalpy flow terms can be rewritten as:

$$\begin{aligned} - \sum_{j=1}^{N_\phi} \int_{S_{CV,j}} \rho^{(da)} \{c_p^{(da)} \mathcal{G} + x \cdot [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \mathcal{G}]\} \mathbf{v} \cdot \mathbf{n} dS = \\ \equiv \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left\{ c_p^{(da)} \langle \mathcal{G} \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} [\Delta h_{l \rightarrow w}^{(wTP)} + \right. \\ \left. + c_p^{(w)} \langle \mathcal{G} \rangle_{S_{CV,j}}] \right\} - \sum_{j=1}^{N_{\phi, out}} \dot{m}_{in_j}^{(da)} \left\{ c_p^{(da)} \langle \mathcal{G} \rangle_{S_{CV,j}} + \right. \\ \left. + \langle x \rangle_{S_{CV,j}} [\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \langle \mathcal{G} \rangle_{S_{CV,j}}] \right\} \end{aligned} \quad (66)$$

Substituting the equalities given by equations (62), (63) and (66) into equation (60) the integral enthalpy balance becomes:

$$\begin{aligned} V_{CV} c_p^{(ma)} \{ \langle x \rangle_{CV} \} \frac{d}{dt} \langle \rho^{(da)} \mathcal{G} \rangle_{CV} + \\ + V_{CV} \left[\Delta h_{l \rightarrow w}^{(wTP)} \frac{d}{dt} \langle \rho^{(da)} x \rangle_{CV} + \right. \\ \left. + c_p^{(w)} \langle \rho^{(da)} \mathcal{G} \rangle_{CV} \frac{d}{dt} \langle x \rangle_{CV} \right] = \\ = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left\{ c_p^{(ma)} \{ \langle x \rangle_{S_{CV,j}} \} \langle \mathcal{G} \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} \Delta h_{l \rightarrow w}^{(wTP)} \right\} + \\ - \sum_{j=1}^{N_{\phi, out}} \dot{m}_{in_j}^{(da)} \left\{ c_p^{(ma)} \{ \langle x \rangle_{S_{CV,j}} \} \langle \mathcal{G} \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} \Delta h_{l \rightarrow w}^{(wTP)} \right\} + \\ + \sum_k^{N_\sigma} \dot{M}_k^{(wv)} h_k^{(wv)} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV + \Phi_D \end{aligned} \quad (67)$$

which is a very complex expression.

5.4 Moist Air Internal Energy Balance

The same procedure used for enthalpy can be used for internal energy starting from equation (31) obtaining:

$$\begin{aligned}
 & V_{CV} c_v^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \frac{d}{dt} \langle \rho^{(da)} g \rangle_{CV} + \\
 & \quad + V_{CV} \left[\Delta u_{l \rightarrow w}^{(wTP)} \frac{d}{dt} \langle \rho^{(da)} x \rangle_{CV} + \right. \\
 & \quad \left. + c_v^{(w)} \langle \rho^{(da)} g \rangle_{CV} \frac{d}{dt} \langle x \rangle_{CV} \right] = \\
 & = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left\{ c_v^{(ma)} \left\{ \langle x \rangle_{S_{CV,j}} \right\} \langle g \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} \Delta h_{l \rightarrow w}^{(wTP)} \right\} + \\
 & - \sum_{j=1}^{N_{\phi, out}} \dot{m}_{in_j}^{(da)} \left\{ c_v^{(ma)} \left\{ \langle x \rangle_{S_{CV,j}} \right\} \langle g \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} \Delta u_{l \rightarrow w}^{(wTP)} \right\} + \\
 & \quad + \sum_k^{N_{\sigma}} \dot{M}_k^{(wv)} u_k^{(wv)} + \Phi_T + \dot{W}_{E,c/e} + \Phi_D
 \end{aligned} \tag{68}$$

where

$$c_v^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \equiv \left[c_v^{(da)} + \langle x \rangle_{CV} c_v^{(w)} \right] \tag{69}$$

is the moist air constant volume specific thermal capacity.

6. Simplified Moist Air Balance Equations

The moist air balance equations derived in the previous paragraph are quite complex and different from the normally used balance equations (1) and (2). To achieve similar formulas, simplifying hypotheses will be employed, which also give more insight into the possible application field of such equations.

The first important simplifying hypothesis is that assessing a **very small average mass by volume change** with respect to the time for the dry air still remaining a compressible fluid; i.e.:

[H.1]

$$\begin{aligned}
 \langle \rho(\mathbf{r}, t) \rangle & = \langle \rho_a + \Delta \rho(\mathbf{r}, t) \rangle = \rho_a \left[1 + \frac{\langle \Delta \rho(\mathbf{r}, t) \rangle}{\rho_a} \right] \\
 \frac{\langle \Delta \rho(\mathbf{r}, t) \rangle}{\rho_a} & \ll 1 \quad \Rightarrow \quad \langle \rho(\mathbf{r}, t) \rangle \cong \rho_a
 \end{aligned}$$

where ρ_a is a constant reference value for the dry air mass by volume, \mathbf{r} is the position vector.

The second important simplifying hypothesis is the **full mix hypothesis**; i.e.:

$$[H.2] \quad \langle a \rangle_{S_{out}} \equiv \langle a \rangle_V$$

i.e. each surface averaged quantity advected out of the control volume has the same value of the volume averaged related quantity.

The third simplifying hypothesis is the **negligible dissipation function** hypothesis,

$$[H.3] \quad \Phi_D \cong 0$$

The integral dissipation function is null for inviscid fluid (null dynamic viscosity), but is also vanishing for non-null viscosity when the velocity vector gradient is very small and/or is affecting a very small part of the control volume (see equations (22) and (33)). The surface-to-volume ratio of the zone volume is an index of such condition: high values of S/V as in air ducts do not allow to neglect the dissipation function as instead allow low values of S/V as in rooms.

6.1 Unsteady Dry Air Mass Balance

Under hypothesis [H.1] the unsteady dry air mass balance equation (52) for compressible fluid becomes:

$$\sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)}(t) \cong \sum_{j=1}^{N_{\phi, out}} \dot{m}_{out_j}^{(da)}(t) \tag{70}$$

which looks like the steady state mass balance, but is not. This because:

$$\begin{aligned}
 V_{CV} \frac{d}{dt} \langle \rho(\mathbf{r}, t) \rangle_{CV} & = \rho_a V_{CV} \frac{d}{dt} \left[1 + \frac{\langle \Delta \rho(\mathbf{r}, t) \rangle_{CV}}{\rho_a} \right] \cong \\
 & \cong \rho_a V_{CV} \frac{d}{dt} [1] = 0
 \end{aligned} \tag{71}$$

6.2 Water Vapour Mass Balance

Under hypothesis [H.1] the unsteady water vapour mass balance equation (58) becomes:

$$\begin{aligned}
 V_{CV} \cdot \rho_a \frac{d}{dt} \langle x \rangle_{CV} & = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \langle x \rangle_{S_{in_j}} - \\
 & - \sum_{j=1}^{N_{\phi, out}} \dot{m}_{out_j}^{(da)} \langle x \rangle_{S_{out_j}} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)}
 \end{aligned} \tag{72}$$

and, under hypothesis [H.2], after substitution of equation (70) into (72), it turns out in:

$$V_{CV} \cdot \rho_a \frac{d}{dt} \langle x \rangle_{CV} = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left(\langle x \rangle_{S_{in_j}} - \langle x \rangle_{CV} \right) + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} \quad (73)$$

6.3 Moist Air Enthalpy Balance

Under hypothesis [H.1], [H.2] and [H.3] and using the dry air mass balance equation (70), the moist air enthalpy balance equation (67) becomes:

$$V_{CV} \rho_a c_p^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \frac{d}{dt} \langle \mathcal{G} \rangle_{CV} + V_{CV} \rho_a h^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} \frac{d}{dt} \langle x \rangle_{CV} = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left\{ c_p^{(ma)} \left\{ \langle x \rangle_{S_{CV,j}} \right\} \langle \mathcal{G} \rangle_{S_{CV,j}} + \langle x \rangle_{S_{CV,j}} \Delta h_{l \rightarrow w}^{(wTP)} - c_p^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \langle \mathcal{G} \rangle_{CV} + \langle x \rangle_{CV} \Delta h_{l \rightarrow w}^{(wTP)} \right\} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} h_k^{(wv)} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV \quad (74)$$

where

$$h^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} = \left[\Delta h_{l \rightarrow w}^{(wTP)} + c_p^{(w)} \langle \mathcal{G} \rangle_{CV} \right] \quad (75)$$

is the water vapour specific enthalpy at the volume averaged air temperature.

If in the advective enthalpy flows we add and subtract the term:

$$c_p^{(ma)} \left\{ \langle \mathcal{G} \rangle_{S_{CV,j}} \right\} \langle \mathcal{G} \rangle_{CV} - c_p^{(ma)} \left\{ \langle \mathcal{G} \rangle_{S_{CV,j}} \right\} \langle \mathcal{G} \rangle_{CV} = 0 \quad (76)$$

and split the source terms as it follows:

$$h_k^{(wv)} \left\{ \mathcal{G}_k \right\} = h^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} + c_p^{(w)} \left(\mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \right) \quad (77)$$

equation (74) can be rewritten as:

$$V_{CV} \rho_a c_p^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \frac{d}{dt} \langle \mathcal{G} \rangle_{CV} + h^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} \cdot \left\{ V_{CV} \rho_a \frac{d}{dt} \langle x \rangle_{CV} \right\} = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} \left\{ \langle x \rangle_{S_{CV,j}} \right\} \left(\langle \mathcal{G} \rangle_{S_{CV,j}} - \langle \mathcal{G} \rangle_{CV} \right) + h^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} \cdot \left\{ \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left(\langle x \rangle_{S_{CV,j}} - \langle x \rangle_{CV} \right) \right\} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} \left\{ \right\} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} c_p^{(w)} \left\{ \mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \right\} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV \quad (78)$$

6.4 Moist Air Internal Energy Balance

Using the same hypothesis, [H.1], [H.2] and [H.3] and the dry air mass balance equation (70), and following the procedure undertaken for the enthalpy balance, the internal energy balance equation (69) becomes:

$$V_{CV} \rho_a c_v^{(ma)} \left\{ \langle x \rangle_{CV} \right\} \frac{d}{dt} \langle \mathcal{G} \rangle_{CV} + u^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} \cdot \left\{ V_{CV} \rho_a \frac{d}{dt} \langle x \rangle_{CV} \right\} = \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} c_v^{(ma)} \left\{ \langle x \rangle_{S_{CV,j}} \right\} \left(\langle \mathcal{G} \rangle_{S_{CV,j}} - \langle \mathcal{G} \rangle_{CV} \right) + u^{(wv)} \left\{ \langle \mathcal{G} \rangle_{CV} \right\} \cdot \left\{ \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left(\langle x \rangle_{S_{CV,j}} - \langle x \rangle_{CV} \right) \right\} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} \left\{ \right\} + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} c_v^{(w)} \left\{ \mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \right\} + \Phi_T + \dot{W}_{R,c/e} \quad (79)$$

6.5 Discussion

We can see, both from the enthalpy and the internal energy balance equations, how the moist air energy balance is constituted of two different contributions: one, "thermal", due to an increase/decrease of the volume averaged air temperature, the other just due to an increase of the water vapour mass content of the moist air (this

because of the hypothesis of ideal gas mixture). These two contributions can be split into two separated balance equations as we show in the next paragraph.

7. “Sensible” and “Latent” Energy Balance Equations

In the field of Heating, Ventilating, Air Conditioning (HVAC), it is common to use terms like “sensible heat” and “latent heat”, which are moist air enthalpy changes that are referring to psychrometric processes driven respectively by only temperature difference and only humidity difference. In the first case we refer to the thermal energy balance, in the second just to the humidity balance equation, as we show later. Looking at the Moist Air Enthalpy Balance, given by equation (78), and comparing it with the Water Vapour Mass Balance, given by equation (73), it is evident that we can delete the second terms (between braces) on the left and right side of equation (78). In fact, those terms are just the left and the right side of the water vapour mass balance equation times an identical quantity, the water vapour specific enthalpy at volume-averaged temperature. The same concerns the Moist Air Internal Energy Balance. Thus, we have available two alternative **thermal energy balance** equations (or **sensible energy**):

- **enthalpy based:**

$$\begin{aligned} V_{CV} \rho_a c_p^{(ma)} \{ \langle x \rangle_{CV} \} \frac{d}{dt} \langle \mathcal{G} \rangle_{CV} &= \\ &= \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} \{ \langle x \rangle_{S_{CV,j}} \} \left(\langle \mathcal{G} \rangle_{S_{CV,j}} - \langle \mathcal{G} \rangle_{CV} \right) + \\ &+ \sum_k^{N_{\sigma}} \dot{M}_k^{(wv)} c_p^{(w)} \{ \mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \} + \Phi_T + \int_{CV} \frac{DP}{Dt} dV \end{aligned} \quad (80)$$

- **internal energy based:**

$$\begin{aligned} V_{CV} \rho_a c_v^{(ma)} \{ \langle x \rangle_{CV} \} \frac{d}{dt} \langle \mathcal{G} \rangle_{CV} &= \\ &= \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} c_v^{(ma)} \{ \langle x \rangle_{S_{CV,j}} \} \left(\langle \mathcal{G} \rangle_{S_{CV,j}} - \langle \mathcal{G} \rangle_{CV} \right) + \\ &+ \sum_k^{N_{\sigma}} \dot{M}_k^{(wv)} c_v^{(w)} \{ \mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \} + \Phi_T + \dot{W}_{R,c/e} \end{aligned} \quad (81)$$

and **one water vapour mass balance equation**, equation (73), which can be seen as a “latent energy” balance equation if multiplied for the water vapour specific enthalpy or internal energy at volume-averaged temperature. Thus, we can have also two “latent energy” balance equation:

- **enthalpy based:**

$$\begin{aligned} h^{(wv)} \{ \langle \mathcal{G} \rangle_{CV} \} \cdot V_{CV} \rho_a \frac{d}{dt} \langle x \rangle_{CV} &= \\ &= h^{(wv)} \{ \langle \mathcal{G} \rangle_{CV} \} \cdot \left\{ \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left(\langle x \rangle_{S_{CV,j}} - \langle x \rangle_{CV} \right) + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} \right\} \end{aligned} \quad (82)$$

- **internal energy based:**

$$\begin{aligned} u^{(wv)} \{ \langle \mathcal{G} \rangle_{CV} \} \cdot V_{CV} \rho_a \frac{d}{dt} \langle x \rangle_{CV} &= \\ &= u^{(wv)} \{ \langle \mathcal{G} \rangle_{CV} \} \cdot \left\{ \sum_{j=1}^{N_{\phi, in}} \dot{m}_{in_j}^{(da)} \left(\langle x \rangle_{S_{CV,j}} - \langle x \rangle_{CV} \right) + \sum_{k=1}^{N_{\sigma}} \dot{M}_k^{(wv)} \right\} \end{aligned} \quad (83)$$

which eventually have to be used together with their “sensible” counterpart.

7.1 Discussion

What equations are then more suitable for a building zone energy balance?

In the HCAV calculation enthalpy based equations are preferred because enthalpy incorporate the flow work, which is an important quantity for thermodynamic open systems as almost all HVAC systems and the building itself are.

The second point in favour of enthalpy based equations is the consideration that the reversible compression/expansion power:

$$\dot{W}_{R,c/e} \equiv - \int_{CV} P \operatorname{div}(\mathbf{v}) dV = \int_{CV} \left(\frac{P}{\rho} \cdot \frac{D\rho}{Dt} \right) dV \quad (84)$$

is something not so simple to calculate: the velocity and pressure field over the control volume have to be known. The only hypothesis that makes this term disappear is that of incompressible fluid, but, dealing with an ideal gas mixture, we cannot apply it.

The “latent energy” balance is not a problem, because it is only conventional and can be taken back to the unique water vapour mass balance

equation (73). This equation can be then rewritten in the same way as equation (2), just specialising the advective and the source terms.

For the thermal energy balance, despite the choice of using the enthalpy based equation, it is not so simple to recover an expression like equation (1).

In the enthalpy based thermal energy balance there is an extra term due to compressibility: the volume integral of the pressure material derivative:

$$\int_{CV} \frac{DP}{Dt} dV = \int_{CV} \left[\frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P \right] dV \quad (85)$$

Also this integral is not a simple calculation and needs to know the velocity and pressure field over the control volume. Nevertheless, the difference with the reversible compression/expansion power is that the condition under which it vanishes is simple to state and can be achieved on a practical level.

If in space and time isobaric processes occur inside the control volume, the integral is null.

8. The Isobaric Thermal Balance Equation

Starting from the enthalpy based thermal energy balance defined by equation (79), by the use of the following hypotheses:

$$[\text{H.4}] \quad \sum_k^{N_s} \dot{M}_k^{(wv)} c_p^{(w)} \{ \mathcal{G}_k - \langle \mathcal{G} \rangle_{CV} \} \cong 0$$

The overheating of the sources water vapour respect to the volume averaged air temperature is small enough.

$$[\text{H.5}] \quad \text{Isobaric processes} \Rightarrow \int_{CV} \frac{DP}{Dt} dV = 0$$

introducing the following positions:

$$V_{CV} = V_a$$

$$c_p^{(ma)} \{ \langle x \rangle_{CV} \} = c_p^{(ma)}$$

$$\langle \mathcal{G} \rangle_{CV} = \mathcal{G}_a$$

$$\langle \mathcal{G} \rangle_{S_{CV,j}} = \mathcal{G}_{a,in_j}$$

$$N_{\phi,in} = N_{zone} + N_{infiltration} + N_{internal} + N_{system}$$

$$N_s = N_{en,s} + N_{internal} + N_{system}$$

(86)

and the Newton's Law of Cooling equation (47), the isobaric thermal balance equation has the following aspect:

$$\begin{aligned} V_a \rho_a c_p^{(ma)} \frac{d\mathcal{G}_a}{dt} &= \sum_{i=1}^{N_{en,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \\ &+ \sum_{j=1}^{N_{zone}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \sum_{j=1}^{N_{infiltr}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \\ &+ \sum_{i=1}^{N_{int,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \sum_{j=1}^{N_{int}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \\ &+ \sum_{j=1}^{N_{system}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \sum_{i=1}^{N_{sys,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) \end{aligned} \quad (87)$$

or, placing

$$\begin{aligned} \dot{Q}_I^{(CV)} &= \sum_{i=1}^{N_{int,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \sum_{j=1}^{N_{int}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) \\ \dot{Q}_{SYS}^{(CV)} &= \sum_{i=1}^{N_{sys,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \sum_{j=1}^{N_{system}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) \end{aligned} \quad (88)$$

it can be written as

$$\begin{aligned} V_a \rho_a c_p^{(ma)} \frac{d\mathcal{G}_a}{dt} &= \sum_{i=1}^{N_{en,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \\ &+ \sum_{j=1}^{N_{zone}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \sum_{j=1}^{N_{infiltr}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \\ &+ \dot{Q}_I^{(CV)} + \dot{Q}_{SYS}^{(CV)} \end{aligned} \quad (89)$$

an equation which is quite identical to the usual equation (1). The main difference is in the use of the moist air constant pressure thermal capacity instead of the dry air one, which is also in principle a function of the air humidity by mass. However, it is important to note that such a correspondence is possible only by the application of hypotheses [H.4] and [H.5]

8.1 Surprise

The developed equation is based on the Ideal Gas Law; thus from the constitutive equation (3) we can derive the differential equality:

$$d(\rho^{(da)} \mathcal{G}) = d(\rho^{(da)} T) = \frac{M_m^{(da)}}{R} dP \quad (90)$$

Applying to equation (90) the same decomposition used in [H.1], it follows:

$$\rho_a d\mathcal{G} + \rho_a d\left(\frac{\Delta\rho^{(da)}}{\rho_a} \mathcal{G}\right) = \frac{M_m^{(da)}}{R} dP \quad (91)$$

and thus for such a hypothesis (very small mass by volume changes) it turns out in:

$$\rho_a d\mathcal{G} \cong \frac{M_m^{(da)}}{R} dP \quad (92)$$

Applying the volume average operator and dividing for dt, we finally obtain:

$$\rho_a \frac{d\langle\mathcal{G}\rangle_{CV}}{dt} \cong \frac{M_m^{(da)}}{R} \frac{d\langle P\rangle_{CV}}{dt} \quad (93)$$

Equation (93) states that if an isobaric process occurs the left side, i.e. the temperature time derivative, it is null.

That means that the left side term of the isobaric thermal balance equation (89) is approximately zero, and thus the balance equation should be:

$$\begin{aligned} & \sum_{i=1}^{N_{en,s}} A_i h_{cv_i} (\mathcal{G}_{s_i} - \mathcal{G}_a) + \sum_{j=1}^{N_{zone}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \\ & + \sum_{j=1}^{N_{in,thr}} \dot{m}_{in_j}^{(da)} c_p^{(ma)} (\mathcal{G}_{a,in_j} - \mathcal{G}_a) + \dot{Q}_I^{(CV)} + \dot{Q}_{SYS}^{(CV)} \cong 0 \end{aligned} \quad (94)$$

instead of equation (1).

The “sensible energy” storage term (i.e. the temperature driven part of the moist air enthalpy storage term) is vanishing when applying the ideal gas hypothesis because, under the isobaric process assumption, the thermal expansion due to temperature increase results into an air displacement out of the control volume. The enthalpy increase due to temperature increase is balanced by the enthalpy decrease due to control volume mass reduction.

9. Conclusion

In the present work a consistent analytical derivation of the energy balance equation for building zone internal moist air is presented based on the following hypotheses:

compressible fluid with very small average mass by volume changes during the occurring processes; full mix of all properties inside the control volume; negligible dissipation function, i.e. negligible effects of shear stresses on the energy balance;

negligible overheating of the sources water vapour respect to the volume averaged air temperature; isobaric processes.

Under such hypotheses, the resulting “sensible energy” equation looks like the usual employed thermal energy balance equation in the most used building simulation programs. Small marginal differences are located in the constant pressure specific thermal capacities, which are, in the first case, the moist air ones (slightly depending on the humidity content), while in the second the dry air ones.

But, if the ideal gas hypothesis is consistently applied to the first term (i.e. the storage term) of such equation, this vanishes. Thus, the resulting thermal energy balance equation for the air node in a building zone is not anymore a first order differential equation in the air node temperature, but just an algebraic equation.

This is an important result not only for the correctness of the solution, but also because it eliminates stability problems related to its numerical time integration.

10. Nomenclature

Symbols

A	surface area [m ²]
c_p	constant pressure specific thermal capacity [J/(kg K)]
c_v	constant volume specific thermal capacity [J/(kg K)]
D	Deviatoric stress tensor [N/m ²]
e	specific total energy [J/kg]
E	total energy [J]
f	specific force vector by unit of surface (s) or by unit of mass (b)
F	force vector [N]
h	superficial heat transfer coefficient [W/(m ² K)], (cv) subscript, or superficial mass transfer coefficient, [kg _(DA) /(s m ²)], (m) subscript
h⁽ⁱ⁾	specific enthalpy [J/kg]
H	enthalpy [J]
I	unit second order tensor
m	mass [kg]

\dot{m}	mass flow rate [kg/s]
M_m	molar mass [kg/kmol]
$\dot{M}^{(\alpha)}$	source of α phase or humidity [kg/s]
\mathbf{n}	surface normal unitary vector (positive outward)
p	partial pressure [Pa]
P	thermodynamic pressure [Pa]
\mathbf{P}	momentum vector [N·s]
R	universal gas constant [J/(kg K)]
u	specific internal energy [J/kg]
U	internal energy [J]
\mathbf{v}	velocity vector [m/s]
V	zone volume [m ³]
t	time [s]
T	temperature [K]
\mathbf{T}	stress tensor [N/m ²]
x	air humidity by mass [kg _{wv} /kg _{da}]
\dot{w}	specific mechanical power [W/m ³]
\dot{W}	mechanical power [W]
Δh	phase change specific enthalpy [J/kg]
Δu	phase change specific internal energy [J/kg]
ϑ	temperature [°C]
μ	dynamic viscosity [Pa·s]
λ	thermal conductivity [W/(m K)]
$\boldsymbol{\lambda}$	thermal conductivity tensor [W/(m K)]
ρ	volumetric mass [kg/m ³]
$\Phi_{\mathbf{T}}$	heat flow density vector [W/m ²]
$\Phi^{(x)}$	flow through a surface of quantity x
$\Phi^{(P)}$	momentum flow [N]
$\langle \rangle$	integral average operator
∇	nabla [m ⁻¹]

Subscripts/Superscripts

a	zone indoor air
b	body
c/e	compression/expansion
cv	convective
CV	control volume
D	dissipative
en	envelope
ext	External
da	dry air
fr	fluid reference
I	internal gains
in	inlet

inf	infiltration
k	kinetic
l	liquid
lw	liquid water
m	moisture
ma	moist air
n	normal
N	number
out	outlet
R	reversible
s, S	surface
SYS	heating and/or cooling system
T	thermal
TP	triple point
v	vapour
w	water
wv	water vapour
σ	water vapour mass source
ϕ	advective flows

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