A SUBSTRUCTURING APPROACH TO 3D CONDUCTION PROBLEMS. APPLICATIONS TO BUILDINGS' COMPONENTS.

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ABSTRACT

The present paper describes how the use of a modal synthesis method in the study of the thermal behaviour of buildings both facilitates and hastes the modelling step concerning the problem of linear conduction in multidimensional components. This work is divided into three sections: we first introduce the substructuring approach applied to modal formulation; we then formally describe, through a simplified bidimensional example, the synthesis step leading to a global evolution model computed by substructuration. Supported by tools developped at GISE and illustrating our results with thermograms, we then compare the results obtained by this method (based on a finite element discretization) with those of a reference computed directly (temperature field and heat flows). Finally, a larger problem of a 3D thermal bridge is treated to emphasize the relevance of the method in terms of computational resources when dealing with average-model computer: CPU time but also memory limitation.

INTRODUCTION

The modal synthesis method, issued from mechanical engineering [1] [4], consists of describing a system in terms of simple interconnected elements. Local computations are made on each element independently from the global structure: different local functions are used to characterize the thermal behaviour of each component and to realize the linking-up procedure for each of the coupled interfaces. Modal reduction techniques can significantly limit the number of local terms involved in the afterward synthesis procedure, allowing us to access easily to a reasonably efficient representation of the whole transient model.

The substructuring method applied to thermal problems, using a modal formulation, has been introduced in the nineties. The feasibility and the relevance of this approach was demonstrated a few years ago by the work of [Neveu/Chen/Blanc/Flament] [2] [3]. Since then, improvements have been achieved, first **allowing a generalized coupling method**, and second, making possible **the computation of both the dynamic and the static part** of the thermal system. Its application to 2D or 3D objects rises many problems requiring specific implements that we will describe further along.

AIM OF THE METHOD

Let us briefly recall the classic modal approach [9] [5]. The searched temperature field over a domain Ω is solution of the equations:

$$\forall M \in \Omega, \quad \mathcal{L}(T(M,t)) = C(M)\dot{T}(M,t) \quad (1)$$

$$\forall M \in \partial\Omega, \quad \mathcal{B}(T(M,t)) = U(M,t) \quad (2)$$

$\mathcal{L}(.) = \nabla.K(M)\nabla(.)$ is the generalized Laplacian operator and $\mathcal{B}(.) = -K(M)\nabla(.).\vec{n} - h(M)(.)$ is the generalized boundary conditions operator. Equations (1) and (2) lead to the following matrix system (dimension equal to the number of variable nodes):

$$[A]T + [E]U = [C]\dot{T} \quad (3)$$

$[A]$ is the matrix of the conductivities, $[C]$ the matrix of the capacities and $[E]$ the matrix of the

driving forces.

The transformation to the modal formulation induces, thanks to the superposition principle, the separation of the temperature field between a dynamic part $T_d(M,t)$ and a pseudo-static part $T_g(M,t)$ such that $\forall M, \quad T(M,t) = T_d(M,t) + T_g(M,t)$. Then, each part is expressed as a function of elementary solutions:

$$\forall M \in \Omega, \quad T_d(M,t) = \sum_i X_i(t)\mathcal{V}_i(M) \qquad (4)$$

$$\forall M \in \Omega, \quad T_g(M,t) = \sum_j U_j(t)\mathcal{S}_j(M) \qquad (5)$$

or, preferring their matrix representation, we obtain $T_d = [P]X$ and $T_g = [S]U$.

$[P]$ is constituted by the eigenmodes of the spatial heat operator $\mathcal{L}(.)$ defined on the domain with homogeneous boundary conditions (the associated eigenvalues λ_i represents the inverse of the structure time constants τ_i). It constitutes the modal basis, whereas $[S]$ is made of static unitary fields corresponding to the response of the system submitted to unitary amplitude sollicitations.

Typically, the computation of the eigenfunctions $\mathcal{V}_i(M)$ leads to the diagonalization of the pair of symetric matrices $[A]$ and $[C]$ (from equation (3)), and the computation of the unitary fields $\mathcal{S}_j(M)$ is given by the inversion of the sole conductivity matrix $[A]$.

The interest of the substructuring approach is precisely to avoid the cost of computation for both $\mathcal{V}_i(M)$ and $\mathcal{S}_j(M)$ functions. It consists in determining those functions by the mean of local interconnected modal models whose capacity to be reduced significantly lightens the cost in terms of memory storage and CPU time. Another advantage of the method derives from the fact that the local modal models are all previously defined independently from the global structure, developing, in this respect, an entirely modular approach.

THE SUBSTRUCTURING UNSTEADY-STATE MODEL

We illustrate the fundamentals of the method by the study of a simple example [1]. Let us consider the thermal bridge Ω described in Figure 1. It is constituted by two materials: concrete and an insulation layer - their physical properties are given with S.I. units.

Third-kind boundary conditions are prescribed on each boundary in contact with the environment and a Neumann homogeneous boundary condition is prescribed to simulate a monodimensional behaviour away from the corner. The discretization step produces a number of N nodes ($N = 1762$ in our example).

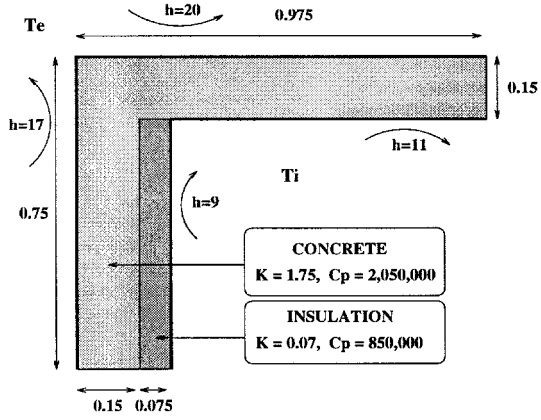

Figure 1: 2D Thermal Bridge Description

Rather than dealing at the same time with the global eigenbasis and the global static fields, let us call $f(M)$ a function defined on the whole domain and let us describe the substructuring method we have developped to obtain an approximation of this function.

1. Decomposition into Components

Decomposition of the domain Ω is the first operation. We choose to divide the structure into three components noted Ω_1, Ω_2 and Ω_3, generating, at the same time, three coupling interfaces I_{kl} (We call I_{kl} the part of boundary of sub-domain k connected with the sub-domain l).

Then, given additional boundary conditions on each coupling interface, we obtain three well-defined local models, for which we will be able to produce a local basis $(W_{ku}(M))_u$ suitable for our modelization. An approximation $f^N(M)$ of the global solution $f(M)$ will then be established throughout its local restrictions $f_k^N(M)$ on each sub-domain as a linear combination of functions issued from the truncated local basis:

$$\forall M \in \Omega_k, \quad f_k^N(M) = \sum_{u=1}^{N_k} c_{ku} W_{ku}(M) \qquad (6)$$

It is now necessary to do the following operations: 1- define the local models, 2- apply a re-

[1] The given results are obtained, with a ultra-sparcstation with 64Mb of RAM, after a discretization step leading to a finite element method involving 8-node quadratic lagrangien elements.

duction technique on those models, 3- exhibit the equations to determine the decomposition coefficients c_{ku}.

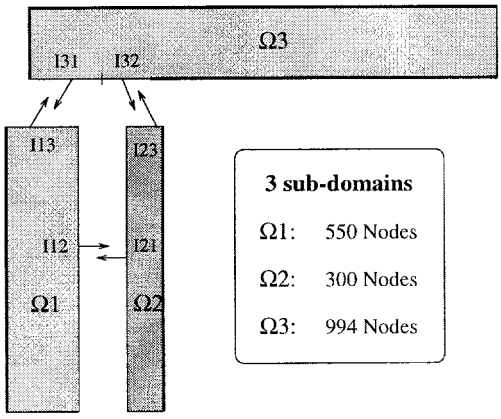

Figure 2: Domain Decomposition of Ω

2. Local Model Computation

The synthesis approach dictates the knowledge, for each sub-domain, of locally defined relevant functions. To this extent, a good idea is to consider successively the local eigenbasis (further called the local modal basis $(V_{kl}(M))_l$ or "transfer basis") and the local unitary fields (further called local static fields $(s_{kj}(M))_j$ defined with Dirichlet-homogeneous boundary conditions for each coupling interface. The computation of the transfer basis needs the diagonalization of a pair of matrices (7) whereas the local static fields only require an inversion (8).

$$[A_k]\,V_{kl} = \lambda_{kl}\,[C_k]\,V_{kl} \quad (7)$$

$$[A_k]\,s_{kj} = e_{kj} \quad (8)$$

$[A_k]$ (resp. $[C_k]$) is the matrix of the conductivities (resp. capacities) specific to the component k. Let us note that all the matrices are symetric with a dimension equal to N_k (number of free nodes on each component).

Unfortunately, we have shown [3] that, as soon as we deal with multidimensional structure, it is not enough to locally reproduce the global behaviour of either the global eigenmodes, or the global static fields, and we must exhibit extra-functions to form a "coupling basis" defined on each interface. These functions are obtained from an eigenvalue problem related to a "local" coupling operator. Its expression is close to the expression of the heat flow exiting an interface:

$$\mathcal{T}_k(Q_k) = -K(M)\frac{\partial \widetilde{Q}_k}{\partial \vec{n}_k} \quad (9)$$

where $\widetilde{Q}_k$ is the harmonic-lifting on Ω_k of Q_k. $\widetilde{Q}_k$ can be seen as a particular temperature field obtained for steady state conditions, with a Dirichlet boundary condition on the interface ($\widetilde{Q}_k = Q_k$) and homogeneous boundary conditions on the other frontiers. The coupling basis computation requires the diagonalization of a couple of symetric matrices but with a very small dimension (equal to the number N_k^Γ of variable nodes on the interface Γ) and use the already computed $[A_k]^{-1}$ to access the harmonic-lifting.

The global function restrictions can now be searched under the following form:

$$\mathcal{V}_{ik}^N = \sum_l p_{ikl} V_{kl} + \sum_q \alpha_{ikq} \widetilde{Q}_{kq} \quad (10)$$

$$\mathcal{S}_{jk}^N = s_j + \sum_q \beta_{jkq} \widetilde{Q}_{kq} \quad (11)$$

Combining all the local functions, we introduce the simplified decomposition formulas: $\mathcal{V}_{ik} = \sum_u c_{iku} W_{ku}^d$ and $\mathcal{S}_{jk} = \sum_u d_{jku} W_{ku}^g$.

For the thermal bridge, approximately 40 minutes are necessary to compute the local models. Table 1 sums up the results in terms of CPU time.

	V_{kl}	$[A]^{-1}$	Q_{kl}	s_{kj}	
Ω_1	228 s	20 s	12 s	2 s	252 s
Ω_2	26 s	2 s	3 s	1 s	32 s
Ω_3	1891 s	204 s	35 s	8 s	2138 s
	2145 s	226 s	50 s	11 s	2432 s

Table 1: CPU time for the local modal models

Previous works [3] authorized only serial linking-up because of the apparition, for a isolated component, of a singularity at the junction of two coupling interfaces, let say Γ_1 and Γ_2. The singularity comes from the necessity to impose there a fixed boundary condition not compatible with boundary condition of the coupling function.

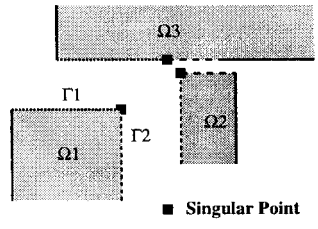

Figure 3: Singular point Example

The desire to keep a modular approach for the computation of the coupling basis prevents us from computing a unique coupling basis defined

on $\Gamma = \Gamma_1 \cup \Gamma_2$. To this end, we decided to add to the coupling basis, for every singular point [2] , a specific singular function equal to 1 on the singularity, 0 on Γ_1 and Γ_2 and harmonic on the subdomain Ω_k. Those "singular functions" won't be discussed any further here. Just keep in mind that, during the computation, they are considered as part of the coupling bases.

3. Local Bases Reduction

The dominance property of the local bases (coupling basis and transfer basis) allows to keep only one part of the functions to reconstitute the global solutions $\mathcal{V}_i$ and $\mathcal{S}_j$.

The currently used criterion to select the dominant local coupling and modal functions is based on their order (Marshall criterion [6]). Previous work shows that the number of local transfer functions to keep depends on the number of global transfer functions to reconstitute, and the number of coupling functions to keep depends on the importance of the coupling interface between components. Some work is now developped at GISE, to find a reduction method well-adapted to the substructuring approach.

For the 2D thermal bridge, we decided to reconstitute the first 100 global eigenmodes. To this extent, we kept around the same amount of local transfer modes for both concrete components. Only 50 modes were conserved for the insulation layer considering its particular dynamic behaviour. All the coupling modes have been kept.

4. Synthesis Procedure

The synthesis process consists in building the equations leading to the computation of the local decomposition coefficients c_{ku} from (6). The restrictions $f_k^N(M)$ they determine must satisfy two "phenomenons": a diffusivity condition imposed on each sub-domain and a continuity condition prescribed on the coupling interfaces.

The former condition is obtained through a localized form analogous to the one verified by the global solution on the system. The second condition represents the coupling conditions between components; in our example, the physical coupling conditions are perfect thermal contact conditions [3], thus the coupling relation represents the continuity of the global functions $\mathcal{V}_i$ or $\mathcal{S}_j$ and the continuity of their fluxes.

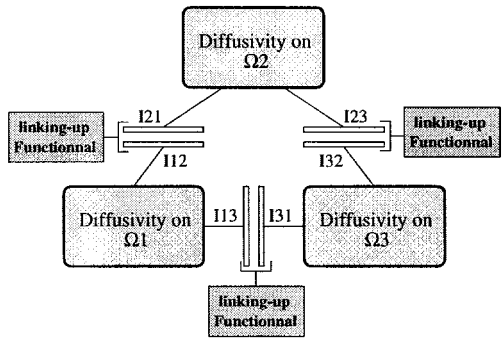

Figure 4: Synthesis Procedure

The global functions computed with the substructuring method do not strictly verify the above coupling conditions, but our method minimizes the shift in the global functions searched, using linking-up functional which satisfy a bilinear symetric form. Both the diffusivity condition and the continuity condition are then transformed into matrix equations:

$$[\mathcal{K}_\mathcal{V}]\, c_i = z_i\, [M]\, c_i \quad (12)$$

$$[\mathcal{K}_\mathcal{S}]\, d_j = e_j \quad (13)$$

The $[\mathcal{K}_\mathcal{V}]$ and $[\mathcal{K}_\mathcal{S}]$ matrices are the modal matrices of the conductivities, used either for the global eigenfunctions or the global static fields. $[M]$ is the modal matrix of the capacities. Their dimension is equal to the number of local modes kept (400 or 160), which ensures a much smaller systems than a directly computed model (1762). z_i is an approximation of the i-th order global eigenvalue λ_i.

Direct Model		
EigenModes :	CPU: 3h10mn	
Static fields :	CPU: 30mn	
Total:	**3h40mn**	
Order:	1762	
SubStructuration Model		
Local models:	CPU: 40mn	Order: 900
EigenModes:	CPU: 6mn	Order: 400
Static fields:	CPU: 2mn	Order: 160
Total:	**48mn**	

Table 2: Computational Resources

Table 2 sums up the results for our models in terms of CPU time computation [4] and the order

[2] Those "singular points" become "singular lines in 3D".

[3] A simple redefinition of the functional would allow us to take into account any contact resistivity between two adjacents components.

[4] This comparison is not totally valuable as we have computed as many eigenmodes as there are freedom degrees - that is 1762 for the direct model and only 400 in the substructuring method.

of the matrices we have to deal with (storage limitation).

The comparison between the eigenvalues of the reference model with the synthesised ones shows a relative error smaller than 0.5%. The first eigenvalue is associated with a time constant equal to $6h24mn22s$.

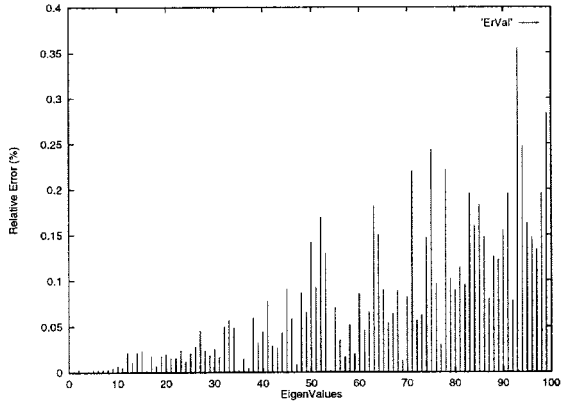

Figure 5: Relative error for the Eigenvalues

To further illustrate the synthesis step, Figure 6 shows the elementary thermogram associated with the second eigenmode of the global system and Figure 7 represents the global unitary field associated to a unitary amplitude temperature applied on the outer walls. The difference with the reference is negligeable.

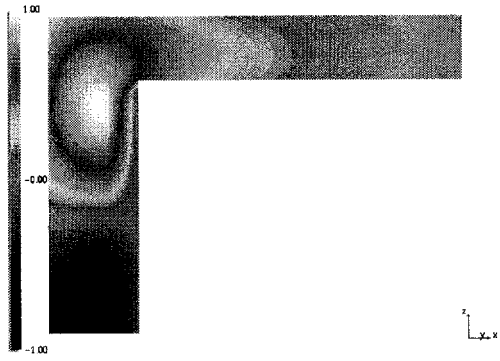

Figure 6: synthesized 2nd EigenFunction

Before reaching the next step, let us point out that once the global model is computed by substructuration, we can, in a way similar to the local models, reduce its size before using it to perform simulation [7] [8].

5. Simulation

Let us take a look at the results obtained during the simulation step. We suppose the structure is

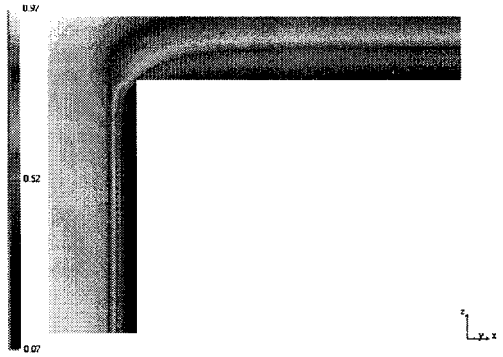

Figure 7: Synthesized Outer Static field

at a constant temperature $T_i = 20°C$ when a step-sollicitation is applied on the outside. The interior is kept at $T_i = 20C$ while the exterior temperature T_e is suddenly set at $0°C$.

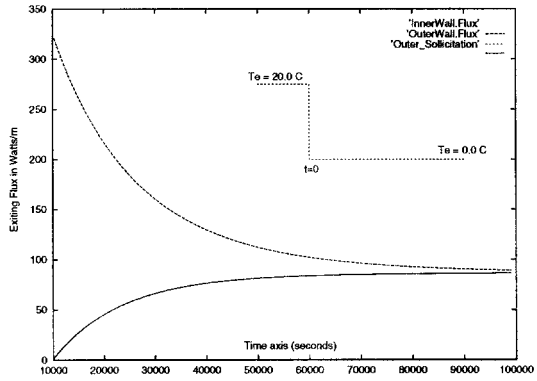

Figure 8: Heat Flow exiting the walls

Figure 8 gives the heat flows exiting the walls for the substructuring model. The steady-state, obtained around 20 hours, shows a loss of 86 Watts.

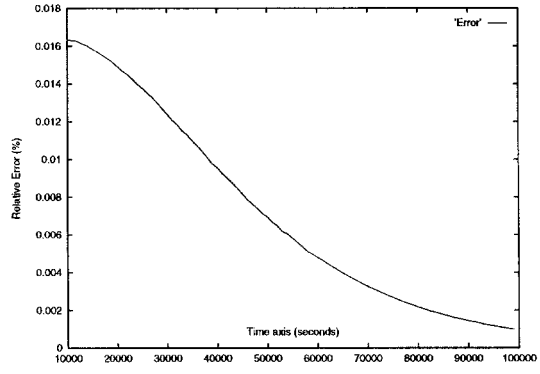

Figure 9: Exiting Heat Flow relative error

Comparison with the reference model shows that the flux values nearly perfectly match. Figure 9 indicates relative errors for the flux exiting the outer walls lower than 0.02%. Knowing from the previous paragraph that the global functions were correctly reconstructed, it is a natural result.

3D EXAMPLE

Now, let us exhibit a tridimensional example which better illustrates the interest of the substructuring approach. Figure 10 presents some part of a room whose walls are in contact with the ground.

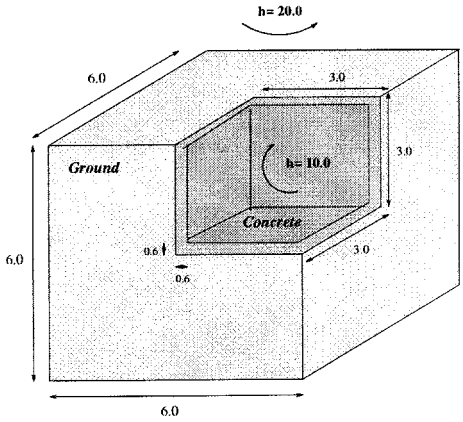

Figure 10: 3D thermal bridge

A discretization step produces around 10.000 nodes, a total which, when dealing with a classical sparcstation, justifies the use of our approach. The decomposition step which follows must be performed keeping in mind the modular aspect of the method. In this respect, we split the system into 8 parts as shown on Figure 11:

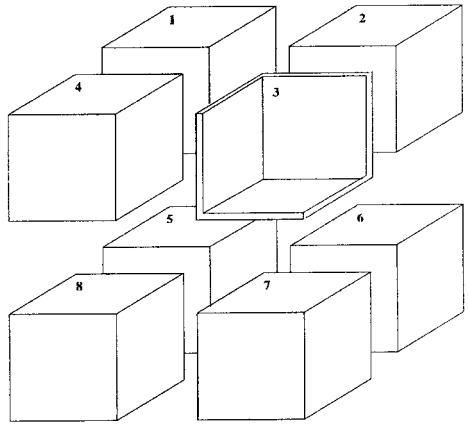

Figure 11: Decomposition

According to the boundary conditions, this choice allows us to consider only three distinct components to build the whole structure: a hollow concrete block (comp. 3) ($K = 1.75$, $C_p = 2,050,000$ (S.I. units)) and two ground blocks ($K = 1.27$, $C_p = 1,440,000$ (S.I. units)); one for the superior level with third-kind boundary conditions (comp. 1, 2 & 4) and one for the inferior level with Neumann boundary conditions (comp. 5, 6, 7 & 8). Computational time required for the local models involved is given in table 3 .

	Nodes	CPU time
Concrete (3)	819	6mn
Ground 1 (1,2,4)	1331	45mn
Ground 2 (5,6,7,8)	1331	45mn

Table 3: Local Models Computation

A total of 1h36mn of CPU time was needed and, given the Dirichlet boundary conditions prescribed on coupling surfaces, the dimension of the higher local model manipulated is equal to 1000. Parallel computations are, of course, possible and even encouraged for this part.

We decide to look for the first 100 global eigenfunctions. To this extent, we build a synthesis model whose order is equal to 2200. 100 transfer modes and half the coupling modes (175) were kept on each component. The full diagonalization process is done in 7 hours. Concerning the static fields substructuration, we have kept a maximum number of local coupling modes. This leads us to a system with 2300 degrees of freedom, which is inversed in 1h30mn [5]. This gives us a total of 8h30mn of CPU time for the synthesised models. However, modifying the algorithm to take into account some kind of Lanczos method would allow us to compute only the first eigenmodes needed for the simulation, reducing the cost. Nevertheless, our main problem was the need in term of memory storage which is here roughly divided by a factor 20 ($= 4.4^2$) obtained on the following comparison:

Direct Model	N = 10136
SubStructuring Model	N = 2300

Figures 14 and 15 respectively exhibit the static field associated with an unitary inner air temperature for the room and the static field associated to an unitary outer air temperature prescribed on ground level. No reference is presented but all the synthesised fields (static or eigenmodes) are correctly reconstructed, the shapes fit well and no bias is visible on the interfaces. We can, now, compute the response of the system to an exterior sinusoidal amplitude temperature $T_e(t) = 5 + 5sin(2\Pi t/24)$, t being given in hours (Figure 12), when the room temperature is set at a constant $20°C$.

Given the previous consideration about the global functions computation, Figure 13 shows the

[5] In this example, most of the order of both synthesised systems is due to the importance of the coupling surfaces. However, the reduction process, even if possible on local coupling basis, is, by now, far from being trivial and is the object of current work.

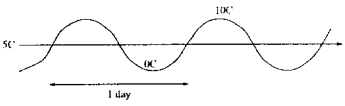

Figure 12: Sinusoidal Amplitude Temperature

consistency of the result in terms of exiting heat flows for the first 10 days of the simulation. Difference in phase is linked with the huge inertia of the system.

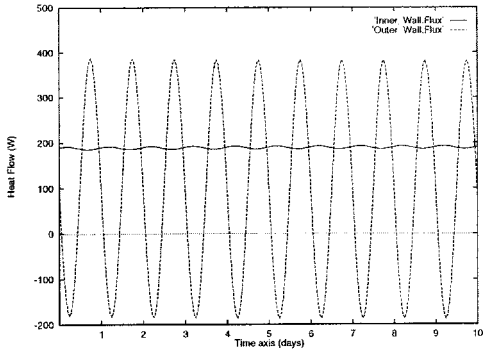

Figure 13: Exiting Heat Flows

CONCLUDING REMARKS

The previous example has shown the relevance of the substructuring method in terms of computational resources and the modelization step can take advantage of its modularity. Nevertheless, even if many improvements could make the present code more efficient, such as optimized diagonalization procedures, the cost is somehow still directly linked to the number of interconnected elements. The next step is to go on developing a hierarchical process ensuring multiple levels of synthesis (thus generalizing what has been done for serial linking [3]). The objective is to integrate all those results in a fully-operating code allowing any kind of linking with different levels of substructuration.

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NOMENCLATURE

General Terms:

$\mathcal{B}(.)$:	generalized boundary conditions
$\mathcal{L}(.)$:	spatial heat operator
M:	space point
t:	time variable
λ_i:	i-th order global eigenvalue

Functions:

$T(M,t)$:	variable temperature field (K)
$T_d(M,t)$:	dynamic temperature field (K)
$T_g(M,t)$:	pseudo-static temperature field (K)
$\mathcal{S}_j(M)$:	j-th order global static field
$\mathcal{V}_i(M)$:	i-th order global eigenfunction
$s_{kj}(M)$:	j-th static field of component k
$Q^{\Gamma}_{kq}(M)$:	q-th coupling mode of component k
$V_{kl}(M)$:	l-th eigenfunction of component k

Matrices and vectors:

A: matrix of the conductivities
C: matrix of the capacities
E: matrix of the driving force
F: matrix of the eigenvalues
P: matrix of the eigenvectors
S: matrix of the static field
U: vector of the inputs
X: state vector

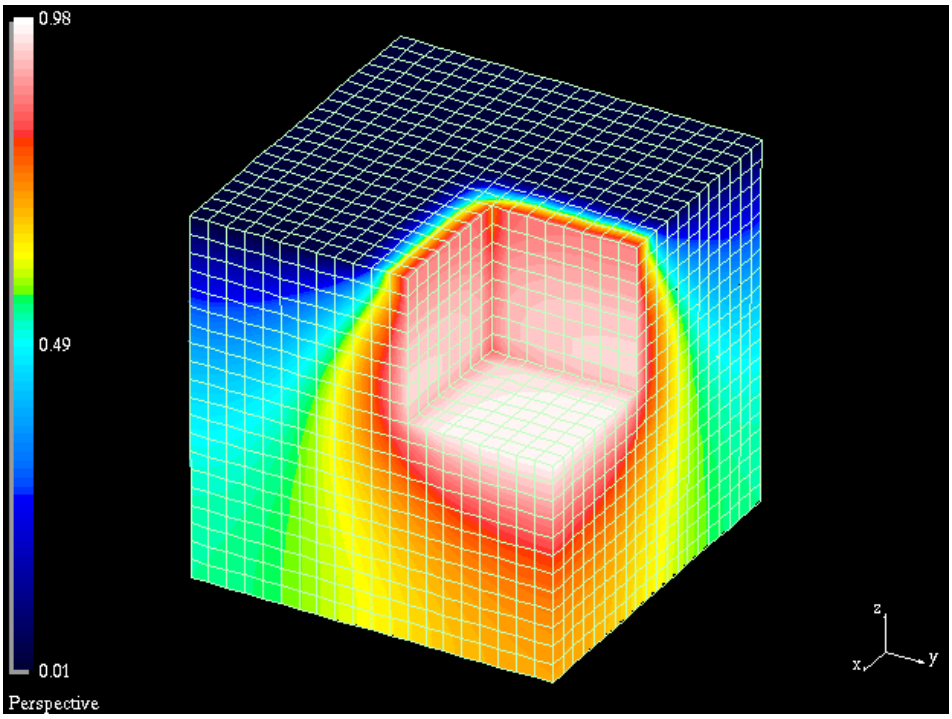

Figure 14: Inner Static Field

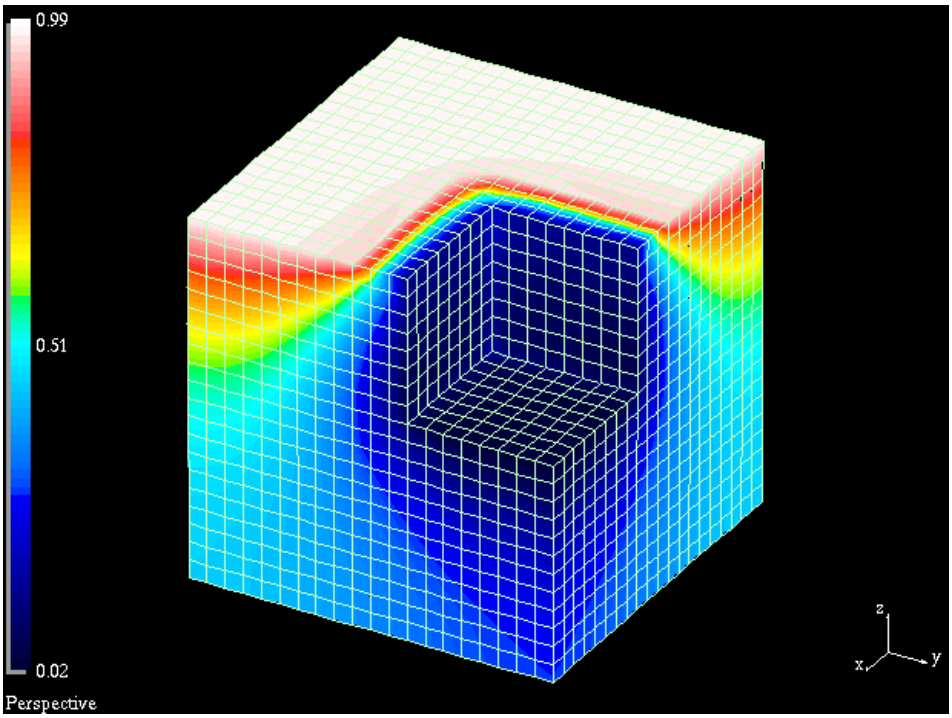

Figure 15: Outer Static Field