GUIDANCE FOR THE SELECTION OF A REDUCTION TECHNIQUE FOR THERMAL MODELS

E. Palomo[*], Y. Bonnefous[*] and F. Déqué[**]
[*] GISE-ENPC, 6-8 av. Blaise Pascal, 77455 Marne la Vallée, France.
[**] EDF-DER, Centre des Renardières B.P. n°1, 77250 Moret sur Loing, France

ABSTRACT

A new method aimed at the selection of the best reduction technique for each given invariant linear system, such as those obtained when modelling the thermal behaviour of building envelopes, is presented here. The method can be divided into three main steps. In the first step, we evaluate *a priori* whether or not the selection of the reduction technique is critical knowing either the desired reduction order or either the level of accuracy required for the reduced model. In the second step, the reduced models are computed with the previously selected criterion and techniques. Finally, the computed reduced model are evaluated *a posteriori* and the best one is selected in the last step. This three step method is implemented in the Matlab Software package, and as been validated at GISE and at EDF.

INTRODUCTION

Nowadays, most of the numerical tools dedicated to modelling the thermal behaviour of buildings, have graphical preprocessors allowing users to quickly built more and more detailed models. The linear systems to be solved for are then larger and larger. And even if computer power keeps increasing, selecting a good model reduction technique is still a critical issue.

In a previous work (Palomo et al., 1995), we have listed and classified the numerous available reduction techniques. However, selecting the most adapted one for a given problem is not an easy task. By looking at the numerical tools involved in each reduction technique we could classify them in terms of required computing time and characteristics of the obtained reduced models. But, there are no general rules for ranking them in terms of accuracy of the obtained reduced models. We decided them to develop a method to easily do this for each given reduction problem.

STATE MODELS FOR LINEAR AND INVARIANT THERMAL SYSTEMS

We are looking at thermal systems which are invariant, linear, and reciprocal. The energy conservation equation after spacial discretisation can be written as:

$$\begin{aligned} C\,\dot{T}(t) &= A\,T(t) + E\,U(t) \\ Y(t) &= J\,T(t) + G\,U(t) \end{aligned} \qquad (1)$$

where T(t) is the vector [N] of the nodal temperatures, C is the matrix [N, N] of the heat capacities at the discretisation nodes, A is the matrix [N, N] of the heat conductances between the internal nodes, E is the matrix [N,p] of the thermal coupling between the system and its environment, and U(t) is the vector [p] of the applied sollicitations. J and G are two matrices [q, N] and [q, p] respectively. In the particular case where Y(t) = T(t), then $J = I_n$ (Identity matrix of order n) and G is the zero matrix.

Let us consider vector X(t) issued from the transformation of T(t) by a regular matrix P:

$$X(t) = P\,X(t)\,; \qquad \det P \neq 0 \qquad (2)$$

Equation (1) becomes then:

$$\begin{aligned} \dot{X}(t) &= P^{-1}\,A\,P\,X(t) + P^{-1}\,E\,U(t) \\ Y(t) &= J\,P\,X(t) + G\,U(t) \end{aligned} \qquad (3)$$

with $A = C^{-1}A$ and $E = C^{-1}E$. Systems (1) and (3) are equivalent, as long as basis transformation matrix P is regular. The most popular basis/transformations for thermal model reductions are the modal basis and the balanced basis. The modal basis is defined by the following eigenvalue problem (cf. Lefebvre, 1987; El Khoury, 1989):

$$C^{-1}A = P\,F\,P^{-1} \qquad (4)$$

where F is a negative definite diagonal matrix composed of the eigenvalues of the matrix $C^{-1}A$ ($\lambda_1, \lambda_2, \ldots, \lambda_n$), and P is the matrix [N, N] of the corresponding eigenvectors placed column wise.

The balanced representation is defined by the following eigenvalue problem (Moore, 1981):

$$W_o W_c = P \Sigma P^{-1} \quad (5)$$

where Σ is a positive definite diagonal matrix made of the Hankel singular values of the system ($\sigma_1^2 \geq \sigma_2^2 \geq ... \geq \sigma_n^2$). W_o and W_c represent the observability grammian and the commandability grammian of the system. They are solutions of the following Lyapunov equations:

$$\begin{aligned} A W_c + W_c A^T + E E^T &= 0 \\ A^T W_o + W_o A + J J^T &= 0 \end{aligned} \quad (6)$$

METHODS TO REDUCE LINEAR AND INVARIANT STATE MODELS

Model reduction has been thoroughly studied and various bibliographic studies have been published, mainly in the automatic application field. Since the eighties, applications in the field of thermal behaviour of solids have been described. Methods are then classified according to the type of reference model (that is the model to be reduced): transfert function, linear state model or non-linear state model. In this paper, we are only looking at invariant linear state models. Such model at generally described by the following equations:

$$\begin{cases} \dot{X}(t) = F X(t) + B U(t), \; X \in \Re^N \\ Y(t) = H X(t) + S U(t) \end{cases} \quad (7)$$

which are fully commandable and observable. A reduced model will be given by:

$$\begin{cases} \dot{\tilde{X}}(t) = \tilde{F} \tilde{X}(t) + \tilde{B} U(t), \; \tilde{X} \in \Re^n, \; n << N \\ \tilde{Y}(t) = \tilde{H} \tilde{X} + \tilde{S} U(t) \end{cases}$$

(8)

and must adequately simulate the behaviour of the reference model (7) when sollicitated by the same inputs U(t). Most of the reduction techniques allow the conservation of the static matrix ($S = \tilde{S}$), which is a key parameter to obtain good reduced models. We will then consider that $S = \tilde{S}$.

Model reduction techniques applied to state linear systems can be divided into three types (Palomo et al., 1995):

- *Minimization techniques*: They look for the best set of ($\tilde{F}$, $\tilde{B}$, $\tilde{H}$) allowing a minimum value for a given relation between Y(t) and $\tilde{Y}(t)$, representing the error induced by the reduction technique. For example, we can cite Wilson technique (Wilson, 1974), which minimize the L_2 norm of the reduction error, and the optimal Hankel norm technique described by Glover (Glover, 1984).
- *Troncation techniques* : They are based on the extraction of a significant part of the reference model. We can cite several troncation techniques in the modal basis (Marshall, 1966; Litz, 1981; Chen, 1989; Ait-Yahia, 1996), and a troncation technique in the balanced basis (Moore, 1981).
- *Mixed techniques*: In the modal basis, they consist in a two step approach. First they look for matrices $\tilde{F}$ and $\tilde{B}$ by a troncation technique and then, they calculate matrix $\tilde{H}$ thanks to a minimization technique (using the L_2 norm mainly). Let us cite for example Michailesco's aggregating technique (Michailesco, 1979) and Oulefki's amalgam technique (Oulefki, 1993).

The main difference between all these techniques and identification techniques, where matrices $\tilde{F}$, $\tilde{B}$ and $\tilde{H}$ would be calculated thanks to numerical minimization procedures of the reduction error for a given set of inputs U(t) is that they are analytical. Matrices $\tilde{F}$, $\tilde{B}$ and $\tilde{H}$ are given as explicit or implicit functions of matrices F, B and H. To do this, a basis transformation and assumptions on the type of inputs are required. Most of the previously cited techniques use the modal or balanced basis and assume Heavyside step function inputs or Dirac impulse function inputs. An analytical expression of the dynamical model response is then obtained.

Once we know the main principle behind each reduction techniques we can easily classify them in terms of required computing time and charateristics of the obtained reduced models. Troncation techniques in the modal basis are the ones which require the less computing time, Marschall reduction technique being the easiest and fastest. Then come the mixed techniques in the modal basis, first Michailesco technique and then Oulefki technique. Moore technique, even if it is a troncation based technique requires more computing time as a transformation to the balanced basis is equivalent to solving for two eigenvalue problems. Glover technique requires also a transformation to the balanced basis. Finally Wilson technique, being iterative, is the one the most expensive in terms of computing time. Concerning the characteristics of the obtained reduced models, first we will notice that all the troncation techniques and most of the mixed techniques in the modal basis, keep some of the eigenelement of the reference model for the reduced model. The time constants of the obtained

reduced models are then a sub-set of the time constants of the reference model. On the other hand, no reference model eigenvalues or singular values are kept in a reduced model obtained by a minimization technique.

Ranking the different techniques in terms of precision of the obtained model is a much more complex problem.

We know that the quality of the reduced model depends on the characteristics of the reference model and on the criterion for evaluating the reduction error and on the spectral characteristics of the inputs. But the diversity in principles, mathematical bases, and norms used in the minimization procedures, makes any theoretical classification impossible in a general case. Then, even if we are only interested in linear invariant models simulating the thermal behaviour of building envelopes, there is not *a priori* a technique better than the other ones. We developped then a procedure to determine for each given reduction problem, *a priori*, whether or not the choice of the reduction technique is an important issue knowing either the desired reduction order or the acceptable maximum reduction error, and then to validate *a posteriori* the obtained reduced model(s).

The thermal model of the ETNA experimental test cell (Girault et al. 1990), obtained with the CLIM2000 software (Rongère et al., 1990), will be used here to illustrate our purpose. This model is of order 68, with three inputs (outside temperature, solar flux and heating power) and one output (inside temperature). The largest time constant of the system is 72 hours and 46 minutes, and the smallest one is 0,25 seconds. Static gains are respectively: 1 for the outside temperature, 0.0144 for the solar flux and 0,0181 for the heating power.

SELECTING THE ORDER AND THE REDUCTION METHOD

We showed that reduction errors can be bounded when using certain error measurement norms and certain types of inputs (Palomo et al., 1995). This will allow us to select *a priori* the best reduction technique(s) knowing either the desired reduction order or the acceptable maximum reduction error.

If $e(t) = Y(t) - \tilde{Y}(t)$ is the reduction error, the equations governing its evolution over time are:

$$\begin{cases} \dot{X}^{\#}(t) = F^{\#} X^{\#}(t) + B^{\#} \dot{U}(t) \\ e(t) = H^{\#} X^{\#}(t) \end{cases} \qquad (9)$$

with

$$X^{\#}(t) = \begin{bmatrix} X(t) \\ \tilde{X}(t) \end{bmatrix}; \; F^{\#} = \begin{bmatrix} F & 0 \\ 0 & \tilde{F} \end{bmatrix}; \quad B^{\#} = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix}; \; H^{\#} = \begin{bmatrix} H & -\tilde{H} \end{bmatrix} \qquad (10)$$

The L_2 norm of the reduction error is given by:

$$\|e(t)\|_2^2 = E\left\{ e^T(t)\, e(t) \right\} \qquad (11)$$

and is representative of the average behaviour of the system. The L_2 norm is generally used when the reduced model will be only used as a prediction tool. However, this norm is not sufficient to qualify the reduced model. We need to have a measure of the magnitude of the variations in the reduction error compared to the magnitude of the variations in the reference model outputs:

$$\eta = \frac{E\left\{ e^T(t)\, e(t) \right\}}{E\left\{ Y^T(t)\, Y(t) \right\}} \qquad (12)$$

The lower bound of the η error measure, for white noise or impulse inputs is then given as a function of the Hankel singular values of the system σ_i ($\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_N$), (Palomo et al., 1995):

$$\eta \geq \frac{\sum_{i=n+1}^{N} \sigma_i^2}{\sum_{i=1}^{N} \sigma_i^2} \qquad (13)$$

This relation is in fact the continuous time version of the relation given by Mirsky (Mirsky, 1960).

For the upper bound, it is obvious that any reduced model of order $n<N$ will satisfy $\eta \leq 1$. However, this upper bound representing a 100% reduction error and which is theoretically associated to an order 0 reduced model, is useless. We selected then the Marshall troncation technique (troncation according to the time constants in the modal basis) to serve as an « upper bound ». The proposed procedure is then:

- Solve for the eigenvalue problem (4) and rank the elements in the modal basis so that $\lambda_1 \le \lambda_2 \le ... \le \lambda_N$. Then apply transformation (2) over (1) to obtain the modal model of the system.
- Compute the grammian W of the states of the modal model. For white noise or impulse inputs, W is solution of the following Lyapunov equation (Arnold, 1974):

  $$F W + W F^T + B B^T = 0$$
- Build matrix $M = H^T H \otimes W$, which defines the L_2 norm of the output vector Y(t) by the sum of its elements. The measure of the error η is then given by (Ait-Yahia, 1996):

$$\eta = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}}{\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}}$$

By a similar procedure, a third useful bound can be calculated: the lower bound for all troncation techniques in the modal basis. In addition, it is generally an upper bound for all the other techniques (mixed techniques in the modal basis and techniques using the balanced basis). This third bound, called medium bound, is defined by the troncation technique proposed by Ait-Yahia (1996), where the elements in the modal basis are re-arranged in order to minimize the L_2 norm of the reduction error.

As a consequence, troncation of the modal reference model and calculation of the Hankel singular values of the system allows us to represent the upper and lower bounds of the L_2 norm of the reduction error as a function of the reduction order.

For the ETNA experimental test cell, the results are given in Figure 1. It shows the different bounds for the L_2 norm of the reduction error normalized by the L_2 norm of the reference model outputs (L_2 normalized) as a function of the reduction order.

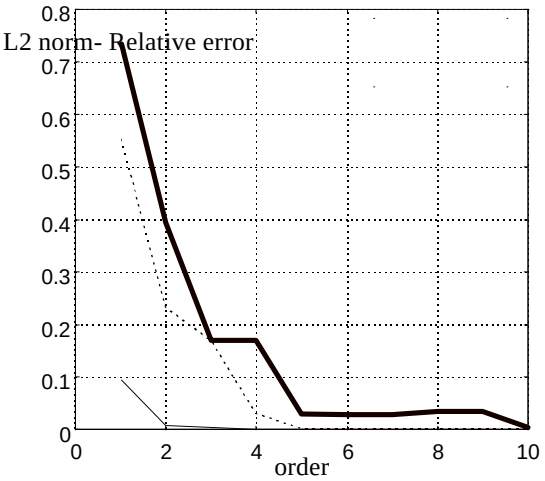

Figure 1. Upper (—), Lower (—) and Medium (---) bounds of the L_2 reduction error norm for the reference model of the ETNA test cell.

By looking at Figure 1, one can conclude that:

1) An optimum L_2 norm reduction technique will allow model reduction to the order 2 or 3.
2) For similar performances in term of induced reduction error, an order 5-10 is required for modal troncation techniques.
3) If the desired reduction order is higher than 4-5, the optimum modal truncation technique, all the mixed techniques and the balanced representation based techniques should be nearly optimal in terms of L_2 norm of the reduction error.

Remind that the reduction error, and its bounds, depends upon the type of sollicitations U(t). Up to now, we have assumed U(t) as a vector made of white noise or impulse signals. However, this bounding procedure can be extended to step input signals. In fact, the fuIl temperature field can be expressed as the sum of a pseudo-static, $T^g(t)$, and a dynamic field, $T^d(t)$:

$$T(t) = T^d(t) + T^g(t) \qquad (14)$$

The pseudo-static term verifies:

$$A\,T^g(t) + E\,U(t) = 0 \qquad (15)$$

and then equation system (1) becomes:

$$\begin{aligned} C\,\dot{T}^d(t) &= A\,T^d(t) + CA^{-1}E\,\dot{U}(t) \\ Y(t) &= J\,T^d(t) + \left[G - JA^{-1}E\right]U(t) \end{aligned} \qquad (16)$$

The bounding procedure, with its basis transformation, is then applied to the equation system (16). We can notice here that assuming

vector U(t) components to be step functions, is equivalent to assuming components of $\dot{U}(t)$ to be white noises or impulses.

EVALUATING REDUCED MODELS

The entire procedure has been implemented in the Matlab programmation environnement. The available reduction techniques are, in the modal basis, Marshall, Michailesco, optimum truncation and Oulefki techniques, and in the balanced basis, Moore and Glover techniques as well as the iterative method of Wilson.

The obtained reduced models are evaluated *a posteriori* in a three step procedure: a) comparison of the induced reduction error with the previously calculated bounds (in terms of L_2 norm of the error); b) analysis of the spectral behaviour of the reduced model and determination of their spectral application domain; and c) analysis of the reduction induced error for a given set of real inputs.

In Figure 2, L_2 reduction error norms are plotted for different reduction techniques and orders (1 to 5). For white noise or impulse inputs U(t), the L2 reduction error norm is given by (Arnold, 1974):

$$F^{\#} W + W F^{\#T} + B^{\#} N B^{\#T} = 0$$
$$\|e(t)\|_2^2 = \text{trace}\left[H^{\#} W H^{\#T}\right] \qquad (17)$$

Figure 2 shows that from the reduction order 3 and up, all the reduction techniques except for modal troncation techniques, are very close to the optimal solution, Moore technique being the best one. From reduction order 5 and up, Moore technique is optimal.

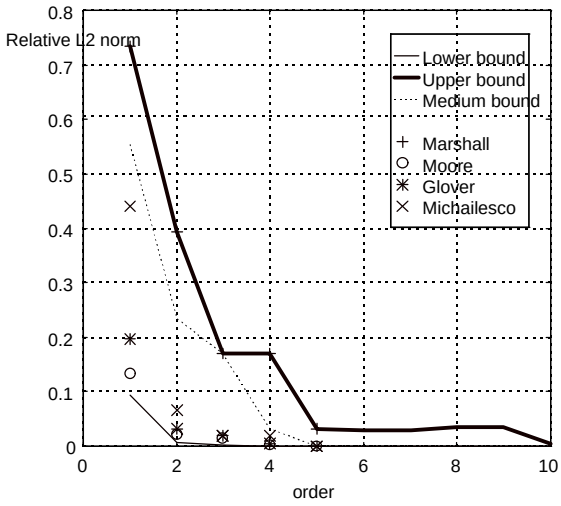

Figure 2. Positioning of the obtained reduced models with the calculated bounds.

After this first step in the evaluation procedure, we go on by comparing the transfer functions of the reference model and of the reduced models, cf. Figure 3 and 4.

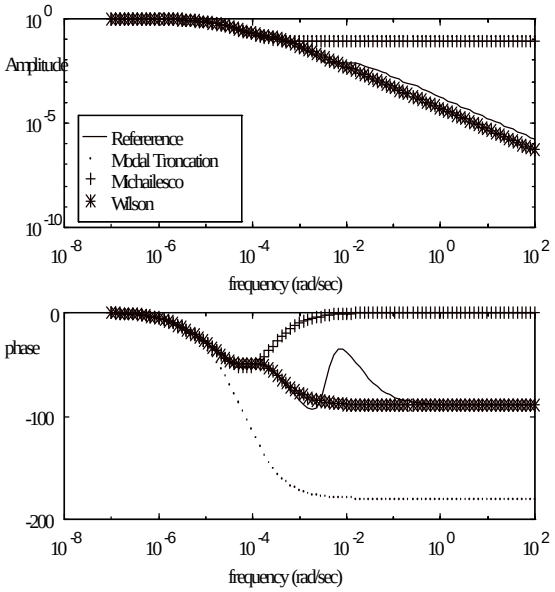

Figure 3. Transfer functions of the reference and 3th order reduced models of the ETNA test cell for the outside temperature input.

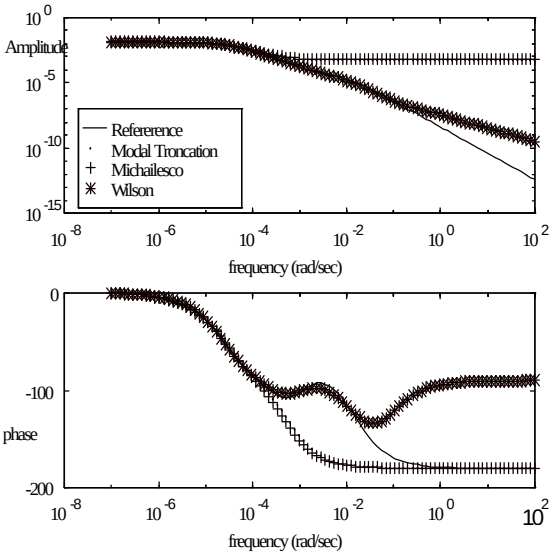

Figure 4. Transfer functions of the reference and 5th order reduced models of the ETNA test cell for the solar flux input.

Figures 3 and 4 shows that all the reduction techniques acts more or less like low frequency band filters. The performances of the reduced models decrease with increasing frequences. Then, we can talk about a spectral application domain as being the frequency domain $[-\omega_{lim}, \omega_{lim}]$ where the

transfert function of the reduced model is an acceptable estimation of the reference model one. This domain can be analytically calculated by:

- Calculating the transfert functions of the reference model and the ones of the reduced model. This is done analytically , while using matrix formulation.

$$\begin{aligned} G(\omega) &= H\left[j\omega I - F\right]^{-1} B + S \\ \tilde{G}(\omega) &= \tilde{H}\left[j\omega I - \tilde{F}\right]^{-1} \tilde{B} + S \\ G^{\#}(\omega) &= G(\omega) - \tilde{G}(\omega) \end{aligned} \qquad (18)$$

- Then, the spectral density of the outputs of the reference model as well as the ones of the reduction error are calculated by using the input spectral matrix and its corresponding transfer function matrix.

$$\begin{aligned} \Gamma_y(\omega) &= G(\omega)\,\Gamma_u(\omega)\,G^{*T}(\omega) \\ \Gamma_e(\omega) &= G^{\#}(\omega)\,\Gamma_u(\omega)\,G^{\#*T}(\omega) \end{aligned} \qquad (19)$$

- The variance of the reference model outputs in the frequency domain [-ω,ω], and the one of the reduction errors, are calculated by integrating the spectral densities given above. If the spectral matrix can be defined analytically, then this calculation is also analytical (Palomo et al., 1996).

$$\begin{aligned} E\left\{ y_i^2(t)\right\}_{-\omega}^{\omega} &= 2\int_{-\omega}^{\omega} \Gamma_{y_i y_i}(\omega)\, d\omega \\ E\left\{ e_i^2(t)\right\}_{-\omega}^{\omega} &= 2\int_{-\omega}^{\omega} \Gamma_{e_i e_i}(\omega)\, d\omega \end{aligned} \qquad (20)$$

- An index of the spectral quality is then given for each output.

$$\eta_i(\omega) = \frac{E\left\{ e_i^2(t)\right\}_{-\omega}^{\omega}}{E\left\{ y_i^2(t)\right\}_{-\omega}^{\omega}} \qquad (21)$$

- Given η_{max} a maximum value or the index previously defined, The spectral application domain of the reduced model is the frequency interval $[-\omega_{lim}, \omega_{lim}]$ such as

$$\forall\, i \quad \eta_i(\omega_{lim}) \le \eta_{max} \qquad (22)$$

For the ETNA test cell model, we get for white noise inputs and η_{max}= 0.01, the following spectral application domains:

Technique	**Order**	**ω_{lim}(rad/s)**
Wilson	5	1.00e+002
Glover	5	3.12e-001
Moore	5	2.52e-001
Wilson	3	1.07e-001
Moore	3	1.94e-002
Glover	3	1.02e-002
Ait-Yahia	5	8.27e-003
Michailesco	5	6.67e-003
Michailesco	3	4.15e-004
Marshall	5	4.15e-004
Ait-Yahia	3	4.90e-005
Marshall	3	2.08e-005

Table 1. Spectral application domains for different reduced models of the ETNA test cell model. The domains are given from the largest to the smallest.

Again, we find that models obtained by modal truncation technique are less performant than the one obtained by Michailesco technique, which is less performant than the ones obtained by Wilson, Moore or Glover technique. For all the techniques, the performance of the obtained reduced model increase with increasing reduction orders. For elevated reduction orders all the reduced models tend to have similar performances. However, for their order 3 reduced models, Wilson, Moore and Glover techniques yield to much larger spectral application domain than the other techniques even for their order 5 reduced models. It is important to notice here that a very small biais in term of L_2 reduction error norm can have a big impact on the quality of the transfer functions for high frequences.

This *a posteriori* evaluation can finally be completed by a standard cross-validation exercice. For a given set of real inputs, we observed the reduction error:

$$e(t) = Y(t) - \tilde{Y}(t)$$

Even if it is time consumming, such a validation exercice is still valuable, and even more when the spectral characteristics of the inputs are not easy to assess.

Figure 5 shows the input signals used for the cross-validation of the reduced models of the ETNA test cell model. They are made out of ROLBS (Randomly Ordered Logarithmicaly Binary Sequence) signals. The maximum standard deviations and maximum reduction errors obtained are given in Table 2.

Figure 6 shows the reduction error for order 5 reduced models obtained by a technique representative of the modal techniques (Michailesco), a technique based on the balanced representation (Moore) and a minimization technique (Wilson).

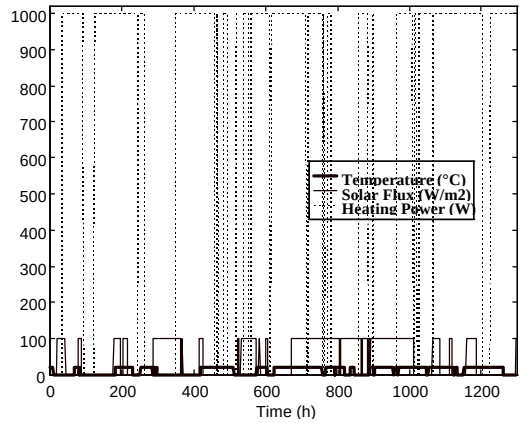

Figure 5. Inputs of the ETNA test cell model.

	Ecart Type (°C^2)		Maximum (°C)	
Méthode	**Ordre 3**	**Ordre 5**	**Ordre 3**	**Ordre 5**
Marshall	2.3637	1.2664	7.8791	6.3765
Micailesco	2.6521	0.8754	8.8132	3.9218
Ait-Yahia	2.3637	0.8909	7.8791	3.9570
Moore	0.3042	0.2889	1.3093	0.9431
Glover	0.8081	0.3195	1.4769	0.8953
Wilson	0.6504	0.1194	1.8215	0.2342

Table 2. Standard deviations and maximum reduction errors associated to reduced models of the ETNA test cell model.

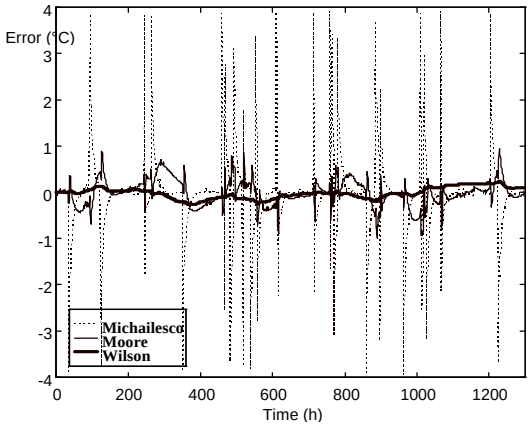

Figure 6. Reduction errors associated with order 5 reduced models of the ETNA test cell obtained by Michailesco, Moore and Wilson techniques.

Ranking the reduced models by the lowest associated standard deviation and up, would lead to the same conclusion as the one drawned from Table 1 (spectral application domain). For this example, Wilson, Moore and Glover techniques give much better results in terms of performance of the obtained reduced models than the modal techniques. Wilson technique giving the best results. In that case, the cross-validation exercice doesn't tell us anything new from what we knew from the previous evaluation steps.

CONCLUSIONS

A full procedure allowing us to select *a priori* one or several reduction technique as well as the reduction orders to explore for a given acceptable reduction error, then calculating the reduced models, and finally evaluating *a posteriori* the obtained reduced model has been developped and implemented in the Matlab programming environnement.

Bounding the L_2 reduction error norm from the calculation of the eigenvalues and the Hankel singular values of the reference model allowed us to elaborate an *a priori* procedure for selecting the most appropriated reduction technique for each given reduction problem, knowing either the desired reduction order or either the acceptable maximum reduction error.

The *a posteriori* evaluation procedure of the obtained reduced models consists into three main steps: a) positioning of the obtained reduced models in the calculated bounds; b) analysis of the spectral quality of the reduced models and calculation of their spectral application domain; c) cross-validation for a given set of real inputs.

The resulting software, called Matred, (Bonnefous et al. 1996) allows any user to easily obtain the best reduced model for his specific problem. This tool is currently being used at the 'Direction des Etudes et Recherches' at EDF. We expect to learn more about its performances as it will now be used by people who are not familiar with reduction techniques and will apply it in other fields.

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