MULTIPLE INPUT TRANSFER FUNCTION MODEL OF GROUND HEAT TRANSFER

JoAnn Amber, P.E.
Mechanical Engineer
U.S.Army CERL
P.O. Box 4005
Champaign, IL 61824-4005

C.O. Pedersen, Ph.D., Professor
Dept of Mech. and Indust. Engineering
University of Illinois
1206 W. Green
Urbana, IL 61801

ABSTRACT

The heat transfer processes occurring in the earth surrounding a building have a substantial effect on the building's energy consumption. During the heating season, for example, heat loss through ground-contact surfaces may be one of the most significant contributors to building heating load. Equipment sizing procedures and building energy analyses must use some method for calculating heat exchange between the building and the surrounding earth if they are to adequately calculate the building heating and cooling loads.

Most existing models of the earth and the earth-building interface are either complex models which require large amounts of computation time or very simple models which are inconsistent with the accuracy of existing detailed hourly building energy calculation programs. BLAST, for example, currently uses a single monthly average ground temperature to define all groundbuilding heat transfer mechanisms. This method does not accurately account for thermal mass effects of the ground beneath the building or the spatial variation of conduction loss due to varying ground temperatures from the surface to the deep ground. Finite element methods can be very accurate, but the required computation time makes them inappropriate for inclusion in existing detailed hourly energy analysis programs.

This paper describes an earth heat transfer model of square slab floors suitable for use with detailed hourly energy programs. The model is based on transfer function methodology using multiple inputs to account for heat transfer between the building, the deep ground and the climate-affected region near ground surface. The transfer functions described in this paper can be calculated from known building and ground characteristics. This model is more accurate than the simple algorithms currently in use while avoiding the excessive computational requirements of more detailed models. The results of this model are compared to a detailed

INTRODUCTION

Existing detailed hourly energy analysis programs such as BLAST do not adequately model the heat transfer between buildings and the ground. Although the model of the building can be very complex, the models of the building-ground heat transfer mechanisms are generally incongruously simple. BLAST, for example, uses a one-dimensional response factor model with a single monthly average ground temperature to define all building-ground heat transfer. A simple model of the ground heat transfer compatible with both existing hourly energy analysis programs and simpler building models becomes more vital as energy conservation techniques reduce the above-ground heat loss and building-ground heat transfer becomes more significant.

Ceylan and Myers [1] developed a response-coefficient method for multidimensional heat conduction problems which is substantially more efficient than finite-difference or finite-element methods. Additionally, it provides a response coefficient model of the system which can be used with any input data which can be approximated by a continuous, piecewise linear function. Seem [2] developed a procedure for calculating multidimensional transfer functions which eliminates some of the computationally expensive steps of the Ceylan and Myers method.

These multidimensional methods have been applied to strictly geometric heat conduction problems. The objective of this study is to extend these techniques from the strictly geometric context of the numerical solution methods to the more conceptual environment of simplified models. Specifically, these concepts will be applied to the problem of heat conduction through slab-on-grade surfaces.

CONCEPT

Many physical systems, including thermodynamic systems, can be approximated using lumped-system analysis. In this approach the system is described as a series of lumped, linear, dynamic elements defined by ordinary differential equations. The network analogy provides a simple visualization of this concept. In a network model of a thermal system, temperatures are represented by nodes with a linear temperature distribution

between each pair of nodes. Physical properties are considered to be uniform between each pair of nodes, but can vary from pair to pair. Energy balance equations are written for each node and the system of equations solved for unknown temperatures and heat fluxes. The validity of the system model is dependent on the accuracy of the assumptions of uniform temperature at each node and linearity between nodes.

Without defining specific geometric or environmental properties, the matrices forming the energy balance equations of the nodes are constructed using state space representation resulting in the state equation

$$\frac{\partial X}{\partial t} = \mathcal{A}X + \mathcal{B}U \qquad (1)$$

and the output equation

$$Q = CX + \mathcal{D}U \qquad (2)$$

The matrix X contains the unknown temperatures (state variables). U is the matrix of known temperatures (input variables). Q is the matrix of fluxes (output variables). Matrices $\mathcal{A}$, $\mathcal{B}$, C, and $\mathcal{D}$ are coefficient matrices. The size of the matrices and the values of the elements will be determined by the specific model. Once the coefficient matrices are defined and the input values identified, the first order differential equations can be solved.

In this study, the method of Seem [2] is used to solve the system of equations. In Seem's formulation, the time series of input variables is modelled as a continuous piecewise linear function. Using this function for the inputs, the differential equations are solved, and substituted for X into Equation (2), resulting in an equation relating the system outputs to the system inputs. This equation is known as the transfer function equation and is of the form

$$Q_t = \sum_{j=0}^{n} (S_j U_{t-j\delta}) - \sum_{j=0}^{n} (e_j Q_{t-j\delta}) \qquad (3)$$

where

Q_t = vector of output variables (heat flux) at time i

S_j = transfer function matrix for temperature inputs at time j

j = designator identifying a point in time, where j=0 is the current time, j=1 is one time step prior to the current time, etc.

t = time of interest

δ = time step

U_i = vector of input variables (known temperatures) at time i

e = scalar constant for adjusting the effect of previous outputs on the output at the time of interest.

STRUCTURE

The system modelled for this study is a square slab-on-grade. The model proposed is the 7 node network with 3 state variables and 4 inputs shown in Figure 1. The known temperatures (or inputs) are the daily average slab core region temperature (T_b), the daily average slab edge region temperature (T_e), the daily average ground surface temperature (T_f) and the deep ground temperature (T_d). The 3 state variables, the temperatures at the remaining nodes (T_1, T_2, T_3), are allowed to float and consequently have some thermal capacitance attributed to them. The temperature nodes are related to each other as shown in the figure. Between attached pairs of temperature nodes, there exists some thermal resistance. The definition of these resistances and capacitances is discussed below.

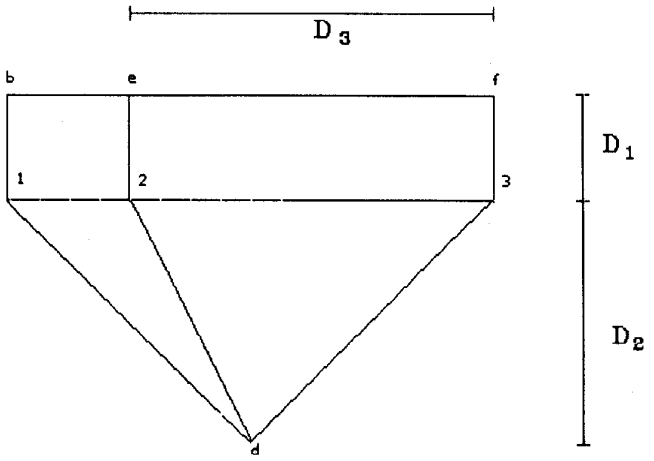

Figure 1: 7 Node Network Model

BASIC EQUATIONS

Energy balance equations are written for each node resulting in 4 state equations of the form

$$C_i \frac{\partial T_i}{\partial t} = \sum_{j=1}^{7} G_{ji}(T_j - T_i) \qquad (4)$$

for i = 1 to 4, and 3 state output equations of the form

$$Q_i = \sum_{j=1}^{7} G_{ji}(T_j - T_i) \qquad (5)$$

for i = 1 to 3, where

C_i = thermal capacitance at node T_i

$G_{ji} = \frac{1}{R_{ji}}$ = inverse of the thermal resistance between nodes T_j and T_i

These 7 equations can be written more conveniently in the form of Equations (1) and (2). In this form the coefficient matrices $\mathcal{A}$, $\mathcal{B}$, C, and $\mathcal{D}$ define the relationships of all temperature regions in the system to all others. They involve geometric factors such as the area through which heat is transferred from one region to another, and physical properties such as the density and thermal conductivity various regions. The purpose of defining the elements of the coefficient matrices is to make it possible to generate transfer function equations for any system from its basic physical parameters rather than as is frequently done in electro-mechanical systems - by testing the system itself. Because the important aspect of the equations is the thermal relationships between regions, the model is not strictly geometric.

The first step in defining the matrix coefficients is to identify the properties which make up the elements of $\boldsymbol{G}$ and $\boldsymbol{C}$. The basic form allows for the description of several heat transfer mechanisms given the appropriate temperatures. For conduction, the equation becomes, in the spatially discretized form used for this model

$$Q_{ij} = \left(\frac{k_{ij} A_{ij}}{L_{ij}}\right)(T_i - T_j). \tag{6}$$

In this case G_{ij} is defined as the conductance, $(k_{ij} A_{ij}) \ / \ L_{ij}$, where

k_{ij} = thermal conductivity applicable to the volume between nodes i and j

A_{ij} = cross-sectional area through which heat is transferred between nodes i and j

L_{ij} = distance between nodes i and j.

The thermal capacitance C is derived from the transient equation

$$Q = \rho c_p V \frac{\partial T}{\partial t} \tag{7}$$

so that

$$C_i = \rho_i c_{p_i} V_i \tag{8}$$

where

ρ_i = density of the region of soil at T_i

c_{p_i} = specific heat of the region of soil at T_i

V_i = volume of the region of soil at T_i.

Both thermal conductance, G_{ij}, and thermal capacitance, C_i, are composed of geometric factors (L_{ij}, A_{ij}, and V_i) as well as soil properties (k_{ij}, ρ_i, and c_{p_i}).

Bahnfleth's study of undisturbed ground temperature patterns [3] shows two distinctly different zones of temperature fluctuation: a relatively fast zone near the ground surface where the temperature changes are in scale with the temperature changes of the forcing temperature, and a slower zone where temperature fluctuations are strongly damped. Because the response rate of the near-surface zone is more similar to the response rate of typical building components than to that of the remainder of the earth, it was decided to model the near-surface earth and the remainder of the earth as attached but distinct components. The point of separation of these zones is the diurnal penetration depth, or roughly 0.5 meters below the surface. The temperature at this point remains nearly constant over a day at the daily average ground surface temperature.

Horizontal maps of ground temperature beneath buildings show a circular pattern, consequently, a cylindrical coordinate system was used to produce an axisymmetric two-dimensional model. Horizontal temperature nodes are set at the slab center, the edge-equivalent radius, and the location where the ground temperature is unaffected by the building (far-field). The edge-equivalent radius is calculated as the radius of a circle of equivalent slab perimeter, or

$$r_p = \frac{P}{2\pi}. \tag{9}$$

Vertical temperature nodes are set at the diurnal penetration depth of the surface temperature wave (approximately 0.5m below the surface), the annual penetration depth (approximately 15m below the surface) and the depth of the point of inflection or knee of the undisturbed temperature profile (approximately 4m below the surface).

The area-equivalent radius is used for calculations in the vertical plane. It is calculated as the radius of the circle which has the same area as the slab, i.e.

$$r_a = \sqrt{\frac{A}{\pi}} \tag{10}$$

so that area is preserved.

Although this model cannot be reproduced graphically, it accounts for both the perimeter and area effects of ground-coupled heat transfer. These geometric relationships were used to build the geometric matrices which were used, in turn, to develop the coefficient matrices. Thermal properties of the soil can be defined separately for each energy balance equation. Individually, these equations assume constant thermal properties. Consequently, the properties are defined as the "effective" value of the thermal properties in the region specified by the equation. The need for an "effective" conductivity is described more thoroughly in Bahnfleth [3].

The inputs to the transfer function equation are (referring to Figure 1) the temperatures of the slab core area (T_b) and the slab edge area (T_e), and the undisturbed ground temperatures near the surface (T_f) and in the deep ground (T_d). Because the "top" nodes are defined at the diurnal penetration depth,

their temperatures can be approximated by the daily average of the surface temperature, i.e. T_b = the daily average slab center temperature and T_f = the daily average ground surface temperature.

For this study, existing data from a one-dimensional semi-infinite solid model of the heat transfer in undisturbed earth [3] were used to provide input data for the nodes at the far-field and the deep ground. The hourly ground surface temperatures calculated by this model vary with local climatic conditions while the deep ground temperature is constant at the annual average ground surface temperature.

Exact data for the daily average slab center and slab edge temperatures were not available from the base case data set. This is the usual case with most energy analysis programs. Therefore, although it is possible in the network mode to include a temperature difference between the slab center and the slab edge, daily average slab surface temperatures were used for both these temperatures. The network parameters developed using the assumption of an isothermal slab should be appropriate for use with energy analysis programs, such as BLAST and DOE-2 which use the same assumption.

Data were available from the base case data set for four locations, Minneapolis MN, Medford OR, Philadelphia PA, and Phoenix AZ. These data were generated originally from TMY weather data for these sites. In order to develop the most responsive model network parameters, the most rigorous weather conditions were used. The weather data for Minneapolis MN would provide the most demanding conditions for the model. In addition to a large annual temperature variation, the temperature variation from day to day is also larger in the Minneapolis data than in that of the other three locations. During the development of the network parameters, therefore, the data derived using Minneapolis weather were utilized to evaluate the accuracy of the model fit. The model was then tested later at the remaining three locations.

GEOMETRIC DEFINITION OF NETWORK PARAMETERS

The geometric definition of the network parameters is based on a series of assumptions about the size and shape of the temperature regions beneath the slab as well as the geometry of the slab itself. The geometry of the slab and the ground in the vicinity of the slab is the foundation for these assumptions. Although the model is not strictly geometric, the heat transfer areas can be initially postulated based on geometric considerations.

First, it is assumed that $A_{1b} = A_{1d}$, $A_{2e} = A_{2d}$, and $A_{3f} = A_{3d}$. Then the area through which heat is transferred between T_b and T_1 can be estimated by the area of the slab which can be approximated by the daily average slab surface center temperature, T_b. Studies [3], [4], and [5] have shown that temperature gradients across horizontal ground-contact surfaces are small over most of the surface and relatively large near the edge.

For this model the area in the plane of the ground surface considered to be at T_b is the area of the slab minus the area near the slab edge, or

$$A_{1b} = A_{1d} = \pi(r_a - d_1)^2 \tag{11}$$

where d_1 is the distance from the slab edge to the location where the slab surface temperature is approximately T_b.

The area through which heat is transferred between T_e and T_2 can be defined as the area where the edge effect predominates. This area is specified as the area within a definable distance, d_2 of the building edge minus the area already associated with T_b, or

$$A_{2e} = A_{2d} = \pi(r_a + d_2)^2 - A_{1b} \tag{12}$$

where d_2 is the distance the edge effect region extends beyond the slab edge. The area associated with the far-field temperature, T_f is calculated from the equation,

$$A_{3f} = A_{3d} = \pi(r_a + D_3)^2 - A_{1b} - A_{2e} \tag{13}$$

where D_3 is the distance from the edge of the slab to the undisturbed ground in the far-field, or 12.5m [3]. The area through which heat is transferred between T_e and T_f is calculated from the Fourier equation for conduction through a hollow cylinder

$$Q = k\left(\frac{2\pi h}{\ln\frac{r_o}{r_i}}\right)\Delta T. \tag{14}$$

By the convention of the matrix definition, all equations are cast in the form of Equation (14). Therefore,

$$Q_{ef} = \frac{k_{ef} A_{ef}}{L_{ef}}(T_e - T_f) \tag{15}$$

so that

$$\frac{2\pi h}{\ln\frac{r_o}{r_i}} = \frac{2\pi L_{2e}}{\ln\frac{L_{be}+L_{ef}}{L_{be}}} = \frac{A_{ef}}{L_{ef}} \tag{16}$$

and,

$$A_{ef} = \frac{2\pi L_{2e} L_{ef}}{\ln\frac{L_{be}+L_{ef}}{L_{be}}} = \frac{2\pi(D_1)(D_3)}{\ln\frac{r_p+D_3}{r_p}}. \tag{17}$$

Similarly,

$$A_{23} = \frac{2\pi L_{2d} L_{23}}{\ln \frac{L_{be} + L_{23}}{L_{12}}} = \frac{2\pi (D_2)(D_3)}{\ln \frac{r_p + D_3}{r_p}}. \quad (18)$$

It is postulated that the slab is isothermal to within d_1 of the slab edge. It is therefore estimated that the area through which heat is transferred between T_b and T_e can also be calculated correspondingly:

$$A_{be} = \frac{2\pi L_{1b}(d_1)}{\ln \frac{L_{be}}{L_{be} - d_1}} = \frac{2\pi (D_1)(d_1)}{\ln \frac{r_p}{r_p - d_1}}. \quad (19)$$

Again, A_{12} is calculated in a similar fashion:

$$A_{12} = \frac{2\pi L_{1d}(d_1)}{\ln \frac{L_{12}}{L_{12} - d_1}} = \frac{2\pi (D_2)(d_1)}{\ln \frac{r_p}{r_p - d_1}}. \quad (20)$$

The volumes associated with temperatures T_1, T_2, and T_3 are calculated by

$$V_1 = A_{1b} h_1 \quad (21)$$

$$V_2 = A_{2e} h_2 \quad (22)$$

$$V_3 = A_{3f} h_3. \quad (23)$$

D_1, D_2, and D_3 are defined by temperature profiles of the undisturbed ground and do not change with slab size or shape. Referencing Figure 2, D_1 is the distance between the inflection point and the diurnal penetration depth, D_2 is the distance between the undisturbed deep ground and the inflection point and D_3 is the distance from the edge of the slab to undisturbed ground temperature in the vertical plane. h_1, h_2, and h_3 Are assumed to be equal to each other and equal to half of the depth of the entire system.

Program GTF calculates the system ground transfer functions (GTFs) by Seem's method ([2],[6]). Program QCALC uses the GTFs and the input temperatures with Equation (3) to calculate the daily average heat flux. Although the annual period of the flux curve was nearly correct, its amplitude was much too small. Because the magnitude of the maximum flux was fairly accurate, it is likely that the primary cause of the error is the assumption of too much mass in the system. The network parameters were based on the assumption that the mass associated with the system is composed of three cylinders of equal depth with cross-sectional areas equal to the area through which heat is transferred vertically in their respective regions. The resulting volume vector heavily weights the system mass with the far-field undisturbed ground temperature. The result is that the mass of soil directly beneath the slab has the least effect. This is obviously incorrect. Various changes to the model were introduced to improve its behavior.

PARAMETER REFINEMENT BY EMPIRICAL METHODS

The effect of soil volume on the shape of the annual flux curve, although evident, is not easily described by geometric techniques. It may be possible to improve the accuracy of the model, particularly in cases where the daily average flux per unit area is small, by adjusting the model parameters based on improved fit to the base case data.

The process of parameter refinement is described in detail in [6]. Both 12m x 12m and 45m x 45m slabs were used to test modifications to the parameter set. The final set of equations for calculating the parameter sets are given below. The quality of the fit of the final models is shown in Table 1. The model is quite accurate for relatively small (144 sq m) to relatively large (2025 sq m) square slabs giving an error in total annual energy consumption of less than 3% in both cases. The model is slightly more accurate overall for the larger slab based on the percentage of the data within 15% of the FDM: 97% for the larger slab vs. 89% for the smaller slab.

FINAL DEFINITION AND TESTING

The geometric matrices were generated were constructed from a series of equations developed using both geometric and empirical methods. The equations are given here in their final forms.

$$D_1 = 4.0m - 0.5m = 3.5m. \quad (24)$$

$$D_2 = 15m - 4m = 11m \quad (25)$$

$$D_3 = 12.5m. \quad (26)$$

$$d_1 = 1.0m \quad (27)$$

$$d_2 = 2.5 + 0.15(\textit{characteristic length}) \quad (28)$$

$$h_1 = h_2 = h_3 = \frac{D_1 + D_2}{2} = 7.25\text{m}. \quad (29)$$

$$L_{1b} = L_{2e} = L_{3f} = D_1 \quad (30)$$

$$L_{1d} = L_{2d} = L_{3d} = D_2 \quad (31)$$

$$L_{fe} = L_{23} = D_3. \quad (32)$$

$$r_p = \frac{P}{2\pi}. \quad (33)$$

$$r_o = \sqrt{\frac{A}{\pi}}. \quad (34)$$

$$A_{be} = \frac{2\pi (D_1)(d_1)}{\ln \frac{r_p}{r_p - d_1}} \quad (35)$$

$$A_{ef} = \frac{2\pi (D_1)(D_3)}{\ln \frac{r_p + D_3}{r_p}} \quad (36)$$

$$A_{12} = \frac{2\pi (D_2)(d_1)}{\ln \frac{r_p}{r_p - d_1}} \quad (37)$$

$$A_{23} = 2\pi L_{23}(4.5)(\textit{characteristic length}) \quad (38)$$

$$A_{1b} = A_{1d} = \pi (r_o - d_1)^2 \quad (39)$$

$$A_{2e} = A_{2d} = \pi (r_o + d_2)^2 - A_{1b} \quad (40)$$

$$A_{3f} = A_{3d} = \pi (r_o + D_3)^2 - A_{1b} - A_{2e} \quad (41)$$

$$V_1 = A_{1b} h_1 \quad (42)$$

$$V_2 = A_{2e} h_2 \quad (43)$$

$$V_3 = A_{3f} h_3 \quad (44)$$

The final set of GTF coefficients and scalar constants calculated using the above equations were used with environmental data for the remaining climates, Medford OR, Philadelphia PA, and Phoenix AZ. Table 1 gives numerical data regarding the accuracy of the models.

For Minneapolis, Medford, and Philadelphia, the difference between the GTF model and the FDM is very nearly zero. In all cases the difference is less than 1 W/m^2 except for a few days at the beginning of the annual cycle. In Phoenix where the annual mean flux is approximately 1.5 W/m^2, an error of less than 1 W/m^2 can create a significant percent error even when the actual value of the error is quite small.

Table 1 Results of GTF Model

LOCATION	SLAB AREA m^2	% OF DATA WITHIN 15% OF FDM [%]	% ERROR IN TOTAL ENERGY CONSUMP [%]
MN	144	89	+0.9
MN	2025	97	-2.9
OR	144	78	-0.2
PA	144	78	-0.6
AZ	144	72	-12.0

UTILIZATION OF THE GTF MODEL FOR ENERGY ANALYSIS

Because of its conceptual similarity to existing energy analysis programs using transfer function models of building components, this model is particularly suitable for incorporation into these programs. When used with these types of hourly energy analysis programs, the ground network would be seen as another zone connected to the conditioned space by an "interzone partition" which would include the slab itself and the top 0.5m of soil. The surface inner and outer temperatures of the "partition" would be the calculated hourly slab surface temperature and the GTF model slab center and edge temperatures (which are equal in this case), respectively. An algorithm for the calculation of ground surface temperatures (such as TEARTH, developed by Bahnfleth) must be included in the processing of weather data in order to provide correct input values for the far-field and deep ground temperatures.

This model could also serve as part of a stand-alone slab heat loss program for situations where the daily average slab surface temperatures are known or can be reasonably approximated. Further studies [6] have demonstrated that substituting a constant indoor air temperature in place of the slab surface temperature gives acceptable results for all locations and both slab sizes. The program would also require an algorithm for the calculation of undisturbed ground surface temperatures.

CONCLUSIONS

A simple multiple-input transfer function model of the heat transfer in the ground beneath a square slab models both relatively small slabs where edge effects are strong and larger slabs whose heat flux is more strongly affected by the flux through the core. The model calculates slab heat flux within 1 W/m^2 at all times and for a range of locations.

The full capability of the model was not tested in this study. Further work to develop a definition of the network parameters based on characteristic length could expand the use of the model to non-square and possibly even non-rectangular surfaces. Testing, and if necessary, modification of the parameter equations to support differential slab core and slab edge temperatures would allow the model to be used more effectively for insulation studies where the placement of the insulation is contingent on the slab to environment temperature difference.

REFERENCES

1. Ceylan, H.T., and Myers, G.E., "Long-Time Solutions to Heat-Conduction Transients with Time-Dependent Inputs," ASME Journal of Heat Transfer, v. 102, Feb 1980.
2. Seem, J.E., MODELING OF HEAT TRANSFER IN BUILDINGS, Ph.D thesis, Univ. of Wisconsin-Madison, 1987.
3. Bahnfleth, W.P., THREE-DIMENSIONAL MODELLING OF HEAT TRANSFER FROM SLAB FLOORS, Ph.D. thesis, Univ. of Illinois at Champaign-Urbana, 1989.
4. Shipp, P.H., THERMAL CHARACTERISTICS OF LARGE EARTH-SHELTERED STRUCTURES, Ph.D. thesis, Univ. of Minnesota, 1979.
5. Kusuda, T., and Bean, J.W., "Simplified Methods for Determining Seasonal Heat Loss from Uninsulated Slab-on-Grade Floors", ASHRAE Transactions, v. 90, Part 1b, 1984.
6. Amber, J.A., MULTIPLE-INPUT TRANSFER FUNCTION MODEL OF HEAT TRANSFER FROM SQUARE SLAB FLOORS, Master's Thesis, Univ. of Illinois at Champaign-Urbana, 1989.